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A constitutive framework of plastically deformed damaged continuum and a formulation using the endochronic concept

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Abstract

The concepts of continuum damage mechanics (CDM) are discussed and a constitutive framework of CDM is proposed for infinitesimal deformation based on the internal variables approach. The framework involves transforming the actual damaged continuum into an equivalent fictitious undamaged continuum. A distinction is made between the state of damage and the damage measure. The development makes use of the concept of 'damage force'. The negative of the damage force is related to the energy required to restore the fictitious undamaged continuum to its undamaged state after each step of deformation and damage.

A set of equation and constraint governs the deformation of the fictitious continuum, while another set of equation and constraint governs the damage behavior. The coupling between the deformation and damage processes is provided for by the damage restoring force concept.

Within the proposed constitutive framework, the endochronic concept has been used to derive explicit constitutive equations. The proposed model has been shown to describe the three-dimensional state of deformation of a cylindrical concrete specimen subjected to uniaxial compression.

1. Introduction

A typical work in continuum damage mechanics (CDM) often involves the damage effect variables in the sense of Kachanov (1958) and the effective stress/effective strain concepts. Consequently, the damage mechanics theories are often derived from the usual constitutive theories by use of effective variables, which take into account the effects of damage, in place of original variables. The thermodynamics framework of CDM often involves either the concept of the strain equivalence postulate (Lemaitre and Chaboche, 1978), the strain-energy equivalence postulate (Sidoroff, 1981; Lemaitre and Chaboche, 1985), or the hypothesis of stress working equivalence

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(Chow and Lu, 1992) along with the concepts of continuum mechanics and irreversible changes in the material internal structure. The microdefects are represented at the macroscopic level by a damage variable. A typical elastoplastic damage theory is based on the generalized damage theory initially proposed by Chaboche (1977) and later by Lemaitre (1985). In the 1985 paper, Lemaitre proposed that the damage energy release rate, i.e., energy release by the system during the damaging process, is related to the elastic strain energy. The damage energy release rate is defined by the thermodynamic force conjugate to damage evolution. This treatment amounts to an uncoupled consideration between plasticity and damage processes.

These concepts of CDM have been proposed by different authors associated with different definitions and theories, and not all of them are needed in one theory. Some of the aforementioned concepts and definitions are not even compatible to each other. One of the purposes of this paper is to discuss the existing concepts of CDM and to propose a CDM constitutive framework using a set of concepts in a unified manner. Further constitutive modeling is required to arrive at explicit constitutive equations for CDM, which may be achieved by use of concept such as damage potential within the proposed constitutive framework. Another approach is to formulate the constitutive equations based on an endochronic concept and still using the same constitutive framework. The second purpose of this paper is to develop such an endochronic formulation for CDM. An example will also be given to illustrate the application of the theory.

In Section 2, the continuity tensor and the anisotropic damage tensor are defined. The concepts of continuity tensor and damage tensor are well-known, but the derivation presented is easy to follow and is different from the existing one. The concepts of gross stress, net-stress, effective stress and damage effect tensor are discussed in Section 3. The thermodynamics constitutive framework for CDM is discussed by use of internal state variables and presented in Section 4. According to this approach, the damage tensor is a measurable quantity which works together with the concept of damage internal variables. The damage internal variables define the state of damage. Concepts such as the ‘damage force’ and the ‘restoring force’ are discussed. In Section 5, the plastic deformation and damage processes are discussed and in Section 6, the constitutive equations and constraints for plastic deformation and damage are presented. The two processes are related by the ‘restoring force’. In this discussion, co-rotational rates are used to account for rotation in the principal axes of damage. Author’s endochronic CDM is briefly summarized in Section 7 and in Section 8, an application related to a cylindrical concrete specimen subjected to axial compressive stress is presented.

2. The anisotropic damage tensor

This work uses a second order tensor as a parameter of damage. The tensor defines the loss of net area of material as in the original work of Kachanov (1958). The presentation of damage tensor D_{ij} in this section follows previous works of Murakami and Ohno (1981) and Betten (1983), in which the damage tensor is constructed using area vectors related to Cauchy’s tetrahedron in a damaged state. In Murakami and Ohno’s anisotropic damage theory of creep, the second rank symmetric damage tensor D_{ij} is derived by representing the effects of microscopic grain-boundary cavities in terms of dyadic product of the unit normal vector to the relevant boundary. On the other hand, in a macroscopic approach, Betten derived the damage tensor from a third order,

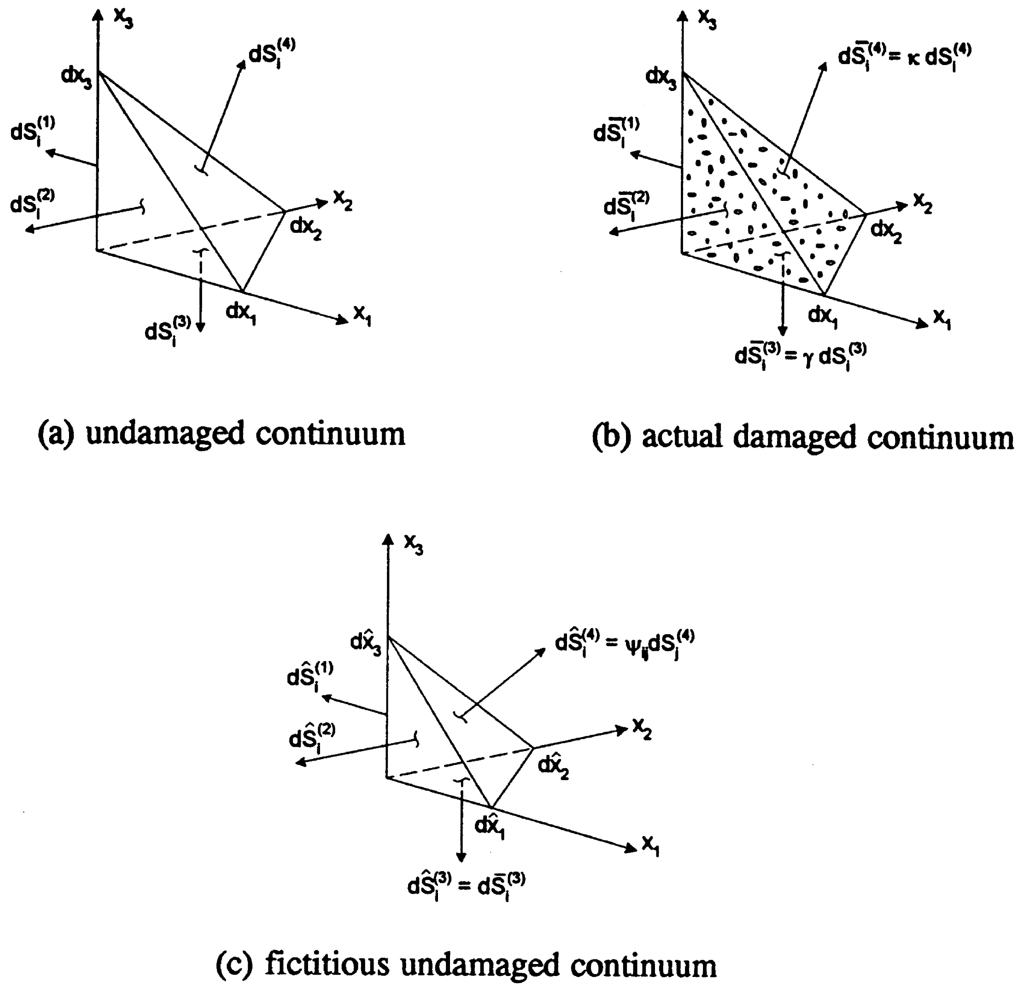

Fig. 1. Definition of the damage measure—the load bearing area.

skew-symmetric, continuity tensor which represents the area vector. In the present work, the derivation of D_{ij} is similar to Betten's derivation, but a second-order continuity tensor is used, which provides a simple and more meaningful physical interpretation.

Consider a differential tetrahedron of an undamaged material as shown in Fig. 1(a). Note that the figure shows a special case where x_i-axes coincide with the principal damage axes. In general, if $\mathrm{d}S_i^{(j)}$ denotes the i-component of a gross area element which has the normal $n_k^{(j)}$, then $\mathrm{d}S_i^{(j)}$ can be characterized by

$$\mathrm{d}S_i^{(1)} = -\tfrac{1}{2}e_{ijk}\,\mathrm{d}x_j^{(2)}\,\mathrm{d}x_k^{(3)}$$

$$\mathrm{d}S_i^{(2)} = -\tfrac{1}{2}e_{ijk}\,\mathrm{d}x_j^{(3)}\,\mathrm{d}x_k^{(1)}$$

$$\mathrm{d}S_i^{(3)} = -\tfrac{1}{2}e_{ijk}\,\mathrm{d}x_j^{(1)}\,\mathrm{d}x_k^{(2)}$$

$$\mathrm{d}S_i^{(4)} = -\tfrac{1}{2}e_{ijk}(\mathrm{d}x_j^{(1)}-\mathrm{d}x_j^{(3)})(\mathrm{d}x_k^{(2)}-\mathrm{d}x_k^{(3)}) \tag{1}$$

where e_{ijk} is the permutation tensor and the vectors $\mathrm{d}x_i^{(j)}$ do not coincide with the principal damage axes in the general case. The sum of these vectors is zero due to closure of the surface area of tetrahedron, i.e.,

$$\mathrm{d}S_i^{(1)}+\mathrm{d}S_i^{(2)}+\mathrm{d}S_i^{(3)}+\mathrm{d}S_i^{(4)} = 0_i \tag{2}$$

Consider now a tetrahedron of a material with internal damage as shown in Fig. 1(b). The nominal dimensions of this tetrahedron are the same as those for Fig. 1(a), but the areas are reduced by scalar factors α, β, γ, and κ from the previous tetrahedron with their corresponding normals unchanged. Then, the area vectors are

$$\begin{aligned}
\mathrm{d}\bar{S}_i^{(1)} &= -\tfrac{1}{2}\alpha_{ijk}\,\mathrm{d}x_j^{(2)}\,\mathrm{d}x_k^{(3)} = \alpha\,\mathrm{d}S_i^{(1)}\\
\mathrm{d}\bar{S}_i^{(2)} &= -\tfrac{1}{2}\beta_{ijk}\,\mathrm{d}x_j^{(3)}\,\mathrm{d}x_k^{(1)} = \beta\,\mathrm{d}S_i^{(2)}\\
\mathrm{d}\bar{S}_i^{(3)} &= -\tfrac{1}{2}\gamma_{ijk}\,\mathrm{d}x_j^{(1)}\,\mathrm{d}x_k^{(2)} = \gamma\,\mathrm{d}S_i^{(3)}\\
\mathrm{d}\bar{S}_i^{(4)} &= -\tfrac{1}{2}\kappa_{ijk}(\mathrm{d}x_j^{(1)}-\mathrm{d}x_j^{(3)})(\mathrm{d}x_k^{(2)}-\mathrm{d}x_k^{(3)}) = \kappa\,\mathrm{d}S_i^{(4)}
\end{aligned} \tag{3}$$

where $\alpha_{ijk} = \alpha e_{ijk}$, $\beta_{ijk} = \beta e_{ijk}$, $\gamma_{ijk} = \gamma e_{ijk}$ and $\kappa_{ijk} = \kappa e_{ijk}$. The areas in eqn (3) represent net cross-sectional areas of the element. These are the areas that are effectively resisting loads and are perpendicular to the coordinate axes x_1, x_2, and x_3, respectively. Note that $\mathrm{d}\bar{S}_i^{(4)}$ denotes the inclined side. The parameters α, β and γ will be further discussed later in this section. It is noted that the vectors $\mathrm{d}\bar{S}_i^{(1)},\ldots,\mathrm{d}\bar{S}_i^{(4)}$, defined in eqn (3), and the corresponding $\mathrm{d}S_i^{(1)},\ldots,\mathrm{d}S_i^{(4)}$, defined in eqn (2), differ in length, and the condition of closure cannot be satisfied, i.e.

$$\mathrm{d}\bar{S}_i^{(1)}+\mathrm{d}\bar{S}_i^{(2)}+\mathrm{d}\bar{S}_i^{(3)}+\mathrm{d}\bar{S}_i^{(4)} \neq 0_i \tag{4}$$

except for the case of isotropic damage where $\alpha = \beta = \gamma = \kappa$.

Because of the existence of microcavities in the material, the load-carrying net areas of the damaged continuum, Fig. 1(b), are reduced. It is now postulated that there exists a fictitious undamaged continuum, as shown in Fig. 1(c), which is mechanically equivalent to the damaged continuum. Thus, the damage state is represented by the fictitious undamaged continuum such that

$$\begin{aligned}
\mathrm{d}\hat{S}_i^{(1)} &= -\tfrac{1}{2}e_{ijk}\,\mathrm{d}\hat{x}_j^{(2)}\,\mathrm{d}\hat{x}_k^{(3)} = \mathrm{d}\bar{S}_i^{(1)}\\
\mathrm{d}\hat{S}_i^{(2)} &= -\tfrac{1}{2}e_{ijk}\,\mathrm{d}\hat{x}_j^{(3)}\,\mathrm{d}\hat{x}_k^{(1)} = \mathrm{d}\bar{S}_i^{(2)}\\
\mathrm{d}\hat{S}_i^{(3)} &= -\tfrac{1}{2}e_{ijk}\,\mathrm{d}\hat{x}_j^{(1)}\,\mathrm{d}\hat{x}_k^{(2)} = \mathrm{d}\bar{S}_i^{(3)}\\
\mathrm{d}\hat{S}_i^{(4)} &= -\tfrac{1}{2}e_{ijk}(\mathrm{d}\hat{x}_j^{(1)}-\mathrm{d}\hat{x}_j^{(3)})(\mathrm{d}\hat{x}_k^{(2)}-\mathrm{d}\hat{x}_k^{(3)})
\end{aligned} \tag{5}$$

where $\mathrm{d}\hat{x}_i^{(j)}$ define the fictitious differential tetrahedron. Furthermore, the closure of the fictitious undamaged continuum is assumed to be satisfied, Thus,

$$\mathrm{d}\hat{S}_i^{(1)}+\mathrm{d}\hat{S}_i^{(2)}+\mathrm{d}\hat{S}_i^{(3)}+\mathrm{d}\hat{S}_i^{(4)} = 0_i \tag{6}$$

The three area vectors $\mathrm{d}\hat{S}_i^{(j)}$ in (5) are identical to the vectors $\mathrm{d}\bar{S}_i^{(j)}$ in (3) and are related to the vectors $\mathrm{d}S_i^{(j)}$ in (2) by scalar factors α, β, and γ, respectively. The fourth vectors $\mathrm{d}\hat{S}_i^{(4)}$ and $\mathrm{d}\bar{S}_i^{(4)}$ are different in both magnitude and direction. Since (5) is used in the remaining part of this paper, the parameter κ is not important and will not be further discussed. It is reasonable to assume that $\mathrm{d}\hat{S}_i^{(4)}$ and $\mathrm{d}S_i^{(4)}$ are related by a linear relation

$$\mathrm{d}\hat{S}_i^{(4)} = \psi_{ij}\,\mathrm{d}S_j^{(4)} \tag{7}$$

where ψ_{ij} is a second-order tensor. In eqn (7), $\mathrm{d}\hat{S}_i^{(4)}$ represents the effective load-carrying area of the damaged material and $\mathrm{d}S_j^{(4)}$ is the gross area on the inclined face of the material element. Therefore, tensor ψ_{ij} represents the fraction of $\mathrm{d}S_j^{(4)}$ that can be used to carry load, accounting for the effect of damage. Tensor ψ_{ij} is referred to as the 'continuity tensor', since it describes the continuity state of the material.

The continuity tensor ψ_{ij} can be determined directly from eqn (7). Substituting (1) and (2) into the right-hand-side of eqn (7) and eqns (3), (5) and (6) into the left-hand-side of eqn (7), one has

$$\alpha_{ijk}\,\mathrm{d}x_j^{(2)}\,\mathrm{d}x_k^{(3)} + \beta_{ijk}\,\mathrm{d}x_j^{(3)}\,\mathrm{d}x_k^{(1)} + \gamma_{ijk}\,\mathrm{d}x_j^{(1)}\,\mathrm{d}x_k^{(2)}$$
$$= \psi_{ir}e_{rjk}(\mathrm{d}x_j^{(2)}\,\mathrm{d}x_k^{(3)} + \mathrm{d}x_j^{(3)}\,\mathrm{d}x_k^{(1)} + \mathrm{d}x_j^{(1)}\,\mathrm{d}x_k^{(2)}) \tag{8}$$

If the vectors $\mathrm{d}x_i^{(j)}$ are aligned with the coordinate axes x_i, respectively, then $\mathrm{d}x_i^{(j)} = \delta_{ij}|\mathrm{d}s^j|$ (no sum on j), where $|\mathrm{d}s^j|$ defines the magnitude of the vector $\mathrm{d}x_i^{(j)}$. Then, for $i = 1$, eqn (8) becomes

$$(\psi_{11} - \alpha)e_{123}|\mathrm{d}s^2|\,|\mathrm{d}s^3| + \psi_{12}e_{231}|\mathrm{d}s^3|\,|\mathrm{d}s^1| + \psi_{13}e_{321}|\mathrm{d}s^1|\,|\mathrm{d}s^2| = 0 \tag{9}$$

Since $|\mathrm{d}s^j|$, the magnitudes of $\mathrm{d}x_i^{(j)}$, are independent of each other, they can be independently varied. But, due to the closure assumption, eqn (9) cannot be violated. Therefore, eqn (9) can be satisfied for all values of $|\mathrm{d}s^j|$, if and only if

$$\psi_{11} = \alpha, \quad \psi_{12} = 0, \quad \psi_{13} = 0 \tag{10}$$

Similar discussion may be made for $i = 2$ and 3. Thus, the continuity tensor is obtained to be

$$\psi_{ij} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \tag{11}$$

It is seen that when the x_i-axes are principal damage axes, the continuity tensor ψ_{ij} is in a diagonal form.

In the case of uniaxial tension along the x_1-direction, let s be the total gross cross-sectional area and $\hat{s}$ be the effective area of resistance so that $\hat{s} < s$. In view of eqn (7), vectors $\mathrm{d}\hat{S}_i^{(4)}$ and $\mathrm{d}S_i^{(4)}$ are represented by $[\hat{s}, 0, 0]^T$ and $[s, 0, 0]^T$, respectively, with $n_i^{(4)} = [1, 0, 0]^T$. Then, by use of eqn (11), eqn (7) reduces to

$$\hat{s} = \psi s \tag{12}$$

where $\psi = \psi_{11} = \alpha$. Therefore, $\psi = \hat{s}/s$ represents that fraction of the cross-sectional area which can be used to resist load. When $\psi = 1$, the material is in the virgin state without damage and $\hat{s}$ is identical to s. When $\psi = 0$, the material can no longer resist load, since its effective area of resistance is reduced to zero.

The damage tensor is defined as a complementary (dual) tensor of continuity (Rabotnov, 1969). In other words, the damage tensor represents the fraction of the cross-sectional area that got reduced by microdefects. In the uniaxial loading case, the damage variable can be expressed in terms of the continuity variable ψ as

$$D = \frac{s-\hat{s}}{s} = 1-\psi \tag{13}$$

Thus, $D = 0$ corresponds to the undamaged state and $D = 1$ corresponds to the breaking state of the material. In the multiaxial case, a second-order damage tensor D_{ij} is defined as

$$D_{ij} = \delta_{ij} - \psi_{ij} \tag{14}$$

In the special case, when the x_i axes are also the principle axes of damage, tensor ψ_{ij} is given by (11), and the damage tensor is given by

$$D_{ij} = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix} = \begin{bmatrix} 1-\alpha & 0 & 0 \\ 0 & 1-\beta & 0 \\ 0 & 0 & 1-\gamma \end{bmatrix} \tag{15}$$

It is seen that the principal values D_1, D_2, and D_3 are related to the principal-continuity variables α, β and γ, respectively. These principal values D_i can be measured on the test specimens cut along mutually perpendicular directions x_1, x_2, and x_3, respectively. Alternatively, the continuity tensor is given in terms of the principal values of damage tensor as

$$\psi_{ij} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} 1-D_1 & 0 & 0 \\ 0 & 1-D_2 & 0 \\ 0 & 0 & 1-D_3 \end{bmatrix} \tag{16}$$

where $\alpha = 1-D_1$, $\beta = 1-D_2$, and $\gamma = 1-D_3$.

3. Gross stress, net-stress and effective stress

In the previous section, a definition of damage is derived by introducing a fictitious undamaged continuum which is mechanically equivalent to the actual damaged continuum. In this section, various definitions of stress, such as the gross stress, the net-stress, and the effective stress, are discussed. The gross stress or the Cauchy stress, σ_{ij}, is the stress defined on the actual damaged continuum while the net-stress, $\hat{\sigma}_{ij}$, and the effective stress, $\tilde{\sigma}_{ij}$, are the non-symmetric and symmetric stress, respectively, defined on the fictitious undamaged continuum.

By considering the actual damaged continuum and the fictitious undamaged continuum under the same applied force, the corresponding stresses on the two continua are different, since the stresses are calculated over different cross sectional areas of the continua. If the equilibrium of the

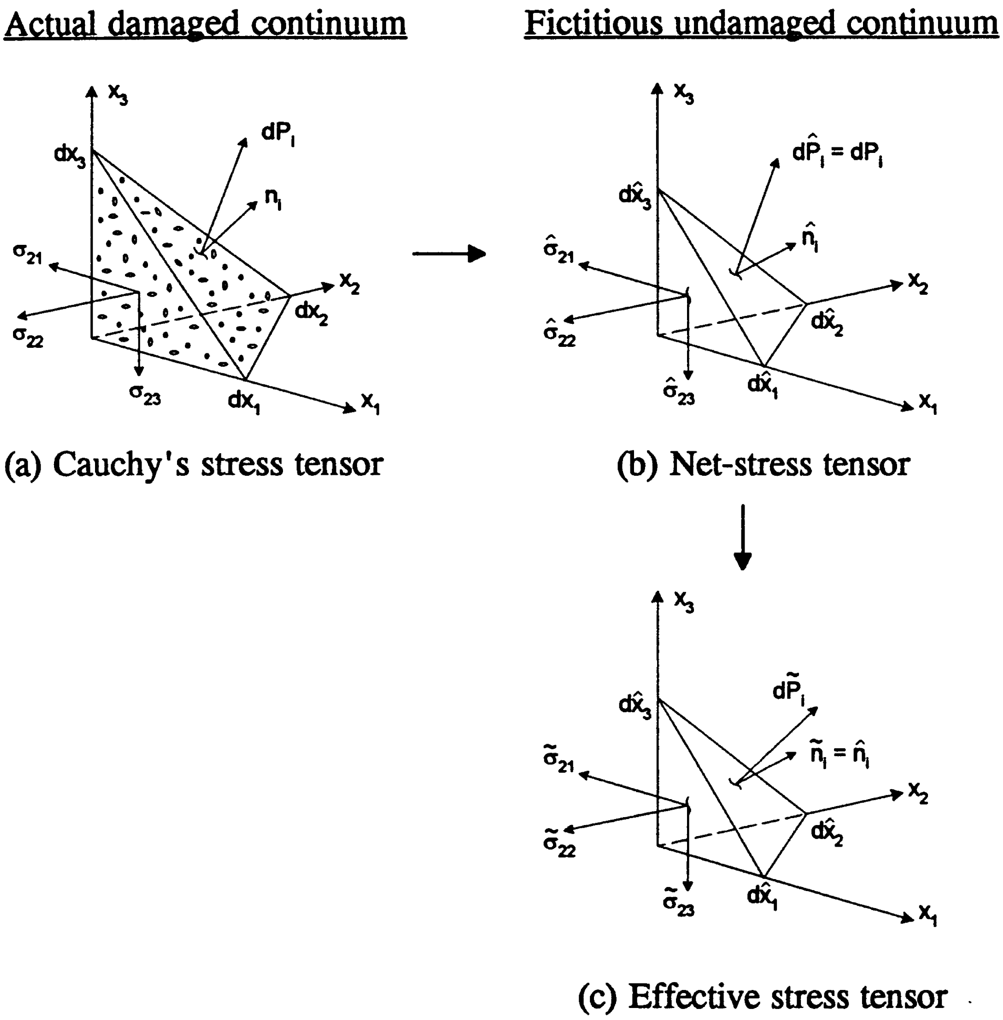

Fig. 2. Definition of stress tensors and the pseudo-force.

actual damaged continuum is considered, Fig. 2(a), one can derive the relation between the stress vector p_i and the stress tensor σ_{ij}, i.e.,

$$p_i = \sigma_{ji} n_j \tag{17}$$

where n_i is the unit normal of an area element $\mathrm{d}S$. Similarly, the equilibrium of the fictitious undamaged continuum, with an area element $\mathrm{d}\hat{S}$ and unit normal $\hat{n}_i$, Fig. 2(b), yields

$$\hat{p}_i = \hat{\sigma}_{ji} \hat{n}_j \tag{18}$$

where $\hat{\sigma}_{ij}$ is the net-stress acted on the fictitious undamaged continuum and $\hat{p}_i$ is the corresponding stress vector. Since the area elements $\mathrm{d}S$ and $\mathrm{d}\hat{S}$ are subjected to the same force, i.e., $\mathrm{d}\hat{P}_i = \mathrm{d}P_i$, one can conclude that

$$\mathrm{d}P_i = p_i\,\mathrm{d}S = \sigma_{ji}n_j\,\mathrm{d}S = \hat{\sigma}_{ki}\hat{n}_k\,\mathrm{d}\hat{S} = \hat{p}_i\,\mathrm{d}\hat{S} = \mathrm{d}\hat{P}_i \tag{19}$$

where $\mathrm{d}S$ and $\mathrm{d}\hat{S}$ are scalar quantities and the corresponding vector expression is obtained from eqn (7) as $\hat{n}_i\,\mathrm{d}\hat{S} = \psi_{ij}n_j\,\mathrm{d}S$. Thus, eqn (19) becomes

$$(\sigma_{ji} - \hat{\sigma}_{ki}\psi_{kj})n_j\,\mathrm{d}S = 0 \tag{20}$$

and it follows that

$$\sigma_{ij} = \psi_{ki}\hat{\sigma}_{kj} \quad \text{and} \quad \hat{\sigma}_{ij} = \psi_{ik}^{-1}\sigma_{kj} \tag{21}$$

By use of eqn (11), the net-stress $\hat{\sigma}_{ij}$ is found to be

$$\hat{\sigma}_{ij} = \begin{bmatrix} \dfrac{\sigma_{11}}{\alpha} & \dfrac{\sigma_{12}}{\alpha} & \dfrac{\sigma_{13}}{\alpha} \\ \dfrac{\sigma_{21}}{\beta} & \dfrac{\sigma_{22}}{\beta} & \dfrac{\sigma_{23}}{\beta} \\ \dfrac{\sigma_{31}}{\gamma} & \dfrac{\sigma_{32}}{\gamma} & \dfrac{\sigma_{33}}{\gamma} \end{bmatrix} \tag{22}$$

which shows that the net-stress $\hat{\sigma}_{ij}$ is non-symmetric, except for the case of isotropic damage. It is not convenient to use the non-symmetric stress tensor $\hat{\sigma}_{ij}$ together with a symmetric strain tensor and/or strain rate in the constitutive equations. Therefore, new symmetrical stress measures, the effective stress $\tilde{\sigma}_{ij}$, have been defined on the fictitious undamaged continuum and used in the constitutive equations. Various definitions have been proposed to symmetrize $\hat{\sigma}_{ij}$. These definitions may be summarized based on various transformations operated on the net stress $\hat{\sigma}_{ij}$. They are:

(a) Betten (1983) proposed a ‘transformed net-stress tensor’, which is an effective stress subjected to the following transformation

$$\tilde{\sigma}_{ij} = \tfrac{1}{2}(\hat{\sigma}_{ij}\psi_{kj}^{-1} + \psi_{ki}^{-1}\hat{\sigma}_{jk}) \tag{23}$$

Using eqn (21), the expression becomes

$$\tilde{\sigma}_{ij} = \tfrac{1}{2}(\psi_{ik}^{-1}\psi_{lj}^{-1} + \psi_{jk}^{-1}\psi_{li}^{-1})\sigma_{kl} = M_{ijkl}\sigma_{kl} \tag{24}$$

where

$$M_{ijkl} = \tfrac{1}{2}(\psi_{ik}^{-1}\psi_{lj}^{-1} + \psi_{jk}^{-1}\psi_{li}^{-1}) \tag{25}$$

The fourth-order transformation tensor M_{ijkl} is referred to as the ‘damage effect tensor’. For ψ_{ij} having the diagonalized form of eqn (11), eqn (24) can be expressed in the matrix form as

$$\begin{Bmatrix} \tilde{\sigma}_{11} \\ \tilde{\sigma}_{22} \\ \tilde{\sigma}_{33} \\ \tilde{\sigma}_{12} \\ \tilde{\sigma}_{23} \\ \tilde{\sigma}_{31} \end{Bmatrix} = \begin{bmatrix} 1/\alpha^2 & 0 & 0 & 0 & 0 & 0 \\ & 1/\beta^2 & 0 & 0 & 0 & 0 \\ & & 1/\gamma^2 & 0 & 0 & 0 \\ & & & 1/\alpha\beta & 0 & 0 \\ & \text{sym} & & & 1/\beta\gamma & 0 \\ & & & & & 1/\gamma\alpha \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} \tag{26}$$

or

$$\tilde{\sigma}_{ij} = \begin{bmatrix} \sigma_{11}/\alpha^2 & \sigma_{12}/\alpha\beta & \sigma_{13}/\alpha\gamma \\ \sigma_{21}/\alpha\beta & \sigma_{22}/\beta^2 & \sigma_{23}/\beta\gamma \\ \sigma_{31}/\alpha\gamma & \sigma_{32}/\beta\gamma & \sigma_{33}/\gamma^2 \end{bmatrix} \tag{27}$$

which is a symmetric stress tensor.

(b) The effective stress proposed by Cordebois and Sidoroff (1979), also by Chow and Wang (1987), is defined through its components given by

$$\tilde{\sigma}_{ij} = \sqrt{\hat{\sigma}_{ij}\hat{\sigma}_{ji}} \quad (\text{no sum on } i \text{ or } j) \tag{28}$$

Since the right-hand-side of (28) is not a tensor operation, the effective stress $\tilde{\sigma}_{ij}$ as defined by eqn (28) is not a tensor. However, in the matrix form, the above definition of effective stress can also give rise to a linear relationship between $\tilde{\sigma}_{ij}$ and σ_{ij}.

(c) In the study of anisotropic damage in the ductile solids Stumvoll and Swoboda (1993) defined the effective stress as the symmetric part of the net-stress tensor, i.e.,

$$\tilde{\sigma}_{ij} = \tfrac{1}{2}(\hat{\sigma}_{ij} + \hat{\sigma}_{ji}) = \tfrac{1}{2}(\psi_{ik}^{-1}\delta_{jl} + \delta_{ik}\psi_{jl}^{-1})\sigma_{kl} \tag{29}$$

where the damage effect tensor is

$$M_{ijkl} = \tfrac{1}{2}(\psi_{ik}^{-1}\delta_{jl} + \delta_{ij}\psi_{jl}^{-1}) \tag{30}$$

By use of eqn (16), M_{ijkl} may be written in terms of the principal damage D_1, D_2 and D_3 and it can be reduced to a form used by Rabotnov (1968) and later by Chow and Lu (1989).

In all cases, the effective stress $\tilde{\sigma}_{ij}$ is related to the Cauchy stress σ_{ij} by the equation

$$\tilde{\sigma}_{ij} = M_{ijkl}\sigma_{kl} \tag{31}$$

where the exact expression for the damage effect tensor M_{ijkl} depends on the method used in symmetrizing $\hat{\sigma}_{ij}$. With respect to the principal damage coordinate system, the damage effect tensor M_{ijkl} is represented by a 6×6 diagonal matrix. In a special case, if the directions of principal stresses coincide with those of the principal damage, then these equations further reduce to

$$\begin{Bmatrix} \tilde{\sigma}_{11} \\ \tilde{\sigma}_{22} \\ \tilde{\sigma}_{33} \end{Bmatrix} = \begin{bmatrix} M_{1111} & 0 & 0 \\ 0 & M_{2222} & 0 \\ 0 & 0 & M_{3333} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{Bmatrix} \tag{32}$$

where M_{1111}, M_{2222}, and M_{3333} are functions of principal damage variables D_1, D_2, and D_3.

The interpretation of the effective stress is now investigated. The net-stress tensor $\hat{\sigma}_{ij}$ is an actual non-symmetric stress acting on the fictitious, undamaged continuum, which is subjected to the same applied force as the original, actual, damaged continuum, i.e., $\mathrm{d}\hat{P}_i = \mathrm{d}P_i$. On the other hand, the effective stress tensor $\tilde{\sigma}_{ij}$ is the fictitious symmetric stress acting on the fictitious undamaged continuum due to the application of the pseudo-force $\mathrm{d}\tilde{P}_i$, as shown in Fig. 2(c). To validate this statement, the Cauchy formula, relating the pseudo-force $\mathrm{d}\tilde{P}_i$ to the effective stress $\tilde{\sigma}_{ij}$ is

$$\mathrm{d}\hat{P}_i = \tilde{\sigma}_{ji}\hat{n}\,\mathrm{d}\hat{S} \tag{33}$$

where $\hat{n}_i$ and $\mathrm{d}\hat{S}$ were previously defined on the fictitious undamaged element. Using eqns (7) and (31), the above relation can be rewritten in terms of the Cauchy stress and the area element $n_i\,\mathrm{d}S$ as

$$\mathrm{d}\tilde{P}_i = M_{jikl}\sigma_{kl}\hat{n}_j\,\mathrm{d}\hat{S} = M_{jikl}\sigma_{kl}\psi_{jm}n_m\,\mathrm{d}S \tag{34}$$

If Betten's definition of damage effect tensor, eqn (25), is used, eqn (34), becomes

$$\mathrm{d}\tilde{P}_i = \psi_{ji}^{-1}\,\mathrm{d}P_j \tag{35}$$

Equation (35) established that the pseudo-force $\mathrm{d}\tilde{P}_i$ is related to the original applied force $\mathrm{d}P_i$ by the inverse-transpose of the continuity tensor ψ_{ij}. If, on the other hand, the damage effect tensor is defined by eqn (30), then the pseudo-force on the fictitious undamaged element is

$$\mathrm{d}\tilde{P}_i = \tfrac{1}{2}(\mathrm{d}P_i + \hat{\sigma}_{ij}\hat{n}_j\,\mathrm{d}\hat{S}) = \tfrac{1}{2}(\mathrm{d}P_i + \psi_{ik}^{-1}\sigma_{jk}\hat{n}_j\,\mathrm{d}\hat{S}) \tag{36}$$

It is noted that $\mathrm{d}\hat{P}_i = \hat{\sigma}_{ji}\hat{n}_j\,\mathrm{d}\hat{S} \neq \hat{\sigma}_{ij}\hat{n}_j\,\mathrm{d}\hat{S}$, due to the non-symmetric property of $\hat{\sigma}_{ij}$. The last term of (36) can be viewed as an additional abstract-force due to the actual stress σ_{ij} acting over the area of the fictitious undamaged continuum, i.e., $\hat{n}_j\,\mathrm{d}\hat{S}$. Therefore, the pseudo-force corresponding to this definition of effective stress has no simple physical interpretation.

4. An internal state variables theory

Based upon concepts of continuum mechanics and irreversible thermodynamics with internal variables, the Clausius–Duhem inequality with respect to the actual, damaged continuum is given by (see Valanis, 1971)

$$\sigma_{ij}\dot{\varepsilon}_{ij} - \dot{\Psi}(\varepsilon_{ij}, q_{ij}^r, D_{ij}, \gamma_{ij}^s, \theta) - \eta\dot{\theta} - \frac{1}{\theta}h_j\theta_{,j} \geqslant 0 \tag{37}$$

In (37), the Helmholtz free energy Ψ is a function of total (elastoplastic) strain ε_{ij}, damage measure D_{ij}, temperature θ, and two sets of internal state variables q_{ij}^r and γ_{ij}^s. There are n number of internal variables q_j^r $(r = 1, 2, \ldots, n)$ which describe the state of plastic deformation and m number of internal variables γ_{ij}^s $(s = 1, 2, \ldots, m)$ which specify the state of damage in the continuum. h_i is heat flux vector and η is entropy density.

In a typical damage mechanics model, the damage tensor D_{ij} is treated as an internal state variable (it is macroscopically not measurable by definition) that describes the irreversible process of internal structure due to microdefects. However, in the present work, the damage tensor D_{ij} is not an internal state variable and it represents a measurable quantity, i.e., the fraction of reduction in load-resisting area. It is a measurable quantity in the description of damage, even though it may be difficult to measure. The role played by D_{ij} in the description of damage is similar to the role played by strain, which is also measurable, in the description of plastic deformation.

In this work, a set of internal state variables γ^s_{ij} is introduced to describe the state of internal damage as a result of growth and/or nucleation of microcracks and/or microvoids. The set of m internal variables γ^s_{ij}, which evolves with loading histories, is introduced to distinguish one internal state of damage from the other, similar to the set of internal variables q^r_{ij} which describes the state of plastic deformation that cannot be uniquely described by the plastic strain alone. The damage variable D_{ij} describes the current fraction of area reduction but not the state of damage. To elaborate, two continua of the same initial damaged state, when undergoing different loading histories, may end up having the same load-resisting area momentarily, hence the same value of D_{ij}, but having two different states of damage.

The concept of using both damage tensor D_{ij} and damage internal state variables γ^s_{ij} in this work is similar but not equal to the concept of Krajcinovic (1985) proposed for the brittle CDM model. In Krajcinovic's model, the microcracks vector fields $\omega^{(i)}$, treated as internal variables, are used to describe the state of damage, and a scalar damage measure D is used to describe the overall damage of the material. However, D is the macroscopic counterpart of the microscopic $\omega^{(i)}$ (they are related by an integral) and D is, therefore, not measurable. In the present work, D_{ij} is defined by a definition not directly related to γ^s_{ij} and it is influenced by the current loading condition. Thus, at the same state of damage, a different incremental loading state will give rise to a different increment of D_{ij}. Hence, $\mathrm{d}D_{ij}$ is different, when the material element is subjected to incremental tension, compression or shear. As an illustration, consider uniaxial tension of a cylinder. The majority of the microcracks will develop in the plane perpendicular to the maximum tensile strain. If the specimen is then unloaded and subsequently subjected to a small compressive stress along its axial direction, the specimen will behave as though it were undamaged up to a certain compressive stress threshold, since all of the microcracks will be passive (crack closure). Consequently, the initial increment of D_{ij} depends on whether the stress increment is tensile or compressive, even though the state of damage is the same at that moment. Furthermore, with the second-order tensor representations of D_{ij} and γ^s_{ij}, the proposed theory is capable of describing both spherical (e.g., void volume fraction) and planar (e.g., a system of planar microcracks) effects, and their interactions, of microcracks.

In the fictitious undamaged configuration, the volume and surface area of the continuum are reduced by excluding the volume and area of the continuum that were previously occupied by microdefects. These are denoted by $\hat{V}$ and $\hat{S}$, respectively. Consequently, the fictitious undamaged matrix material becomes homogeneous and isotropic. For a given force field $\hat{P}_i = P_i$, the first law of thermodynamics written for this fictitious undamaged continuum, is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{V}}\left(\frac{1}{2}\hat{v}_i\hat{v}_i+\hat{u}\right)\hat{\rho}\,\mathrm{d}\hat{V} = \int_{\hat{V}}\hat{\rho}\hat{b}_i\hat{v}_i\,\mathrm{d}\hat{V}+\int_{\hat{S}}(\hat{\sigma}_{ji}\hat{v}_i-\hat{h}_j)\hat{n}_j\,\mathrm{d}\hat{S}+\int_{\hat{V}}\hat{r}\,\mathrm{d}\hat{V} \tag{38}$$

where $(\hat{\ })$ is used to indicate that the quantity is associated with the fictitious undamaged continuum. In (38), $\hat{v}_i$ is the velocity; $\hat{u}$ is the internal energy density; $\hat{\rho}$ is the mass density; $\hat{b}_i$ is the body force; and $\hat{r}$ is the heat source term. The first term in the surface integral represents the rate of work done by surface traction and is expressed in terms of the non-symmetric net-stress tensor $\hat{\sigma}_{ij}$. When the pseudo-force field $\tilde{P}_i$ is introduced to the fictitious continuum so that the corresponding effective stress $\tilde{\sigma}_{ij}$ is symmetric, eqn (38) is written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{V}}\left(\frac{1}{2}\tilde{v}_i\tilde{v}_i+\hat{u}\right)\hat{\rho}\,\mathrm{d}\hat{V}=\int_{\hat{V}}\hat{\rho}\hat{b}_i\tilde{v}_i\,\mathrm{d}\hat{V}+\int_{\hat{S}}(\tilde{\sigma}_{ji}\tilde{v}_i-\hat{h}_j)\hat{n}_j\,\mathrm{d}\hat{S}+\int_{\hat{V}}\hat{r}\,\mathrm{d}\hat{V} \tag{39}$$

Due to the use of pseudo-force field $\tilde{P}_i$, the velocity vector in the configuration is $\tilde{v}_i$ instead of $\hat{v}_i$, as indicated in eqn (39). Consequently, the deformation of the fictitious undamaged continuum subjected to pseudo-force field $\tilde{P}_i$ is different from that subjected to force field P_i. The rate of deformation for the fictitious undamaged configuration is then defined by

$$\dot{\tilde{\varepsilon}}_{ij}=\frac{1}{2}\left(\frac{\partial\tilde{v}_i}{\partial x_j}+\frac{\partial\tilde{v}_j}{\partial x_i}\right) \tag{40}$$

where $\tilde{\varepsilon}_{ij}$ defines the deformation of the fictitious undamaged continuum (with pseudo-force field $\tilde{P}_i$) and is referred to as the effective strain. According to eqn (40), the relationship between the effective strain $\tilde{\varepsilon}_{ij}$ and the actual strain ε_{ij} depends on transformations between velocity vectors from v_i to $\hat{v}_i$ and from $\hat{v}_i$ to $\tilde{v}_i$. In general, the explicit forms of these transformations are difficult to define due to the complexity of the geometry and mathematics involved. In this work, the effective strain $\tilde{\varepsilon}_{ij}$ is expressed in terms of damage tensor D_{ij} and actual strain ε_{ij}, and this relationship will be discussed later in this section.

The postulate of free energy equivalence is applied in the subsequent discussion. According to this postulate, which was initially proposed by Cordebois and Sidoroff (1979) in the form of strain-energy equivalence, the free energy for an actual, damaged material has the same form as that for a fictitious, undamaged material, but the variables are replaced by the effective quantities. Thus,

$$\tilde{\Psi}(\tilde{\varepsilon}_{ij},\tilde{q}^r_{ij},\gamma^s_{ij},\theta)\equiv\Psi(\varepsilon_{ij},q^r_{ij},D_{ij},\gamma^s_{ij},\theta) \tag{41}$$

where $\tilde{q}^r_{ij}$'s are the effective q^r_{ij}'s. Note that D_{ij} does not explicitly appear as one of the state variables on the left-hand-side of (41). In view of eqn (41), the free energy available to do mechanical work and stored in the fictitious continuum is the same as that stored in the actual continuum, resulting in an equivalent mechanical behavior.

The second law of thermodynamics and the equation of motion at the fictitious configuration subjected to the pseudo-force field $\tilde{P}_i$ become

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{V}}\hat{\rho}\eta\,\mathrm{d}\hat{V}\geqslant\int_{\hat{V}}\frac{\hat{r}}{\theta}\mathrm{d}\hat{V}-\int_{\hat{S}}\frac{\hat{h}_i}{\theta}\hat{n}_j\,\mathrm{d}\hat{S} \tag{42}$$

$$\frac{\partial\tilde{\sigma}_{ji}}{\partial x_i}+\hat{\rho}\hat{b}_i=\hat{\rho}\tilde{f}_i \tag{43}$$

where $\tilde{f}_i=(\mathrm{d}\tilde{v}_i/\mathrm{d}t)$. Using eqns (39)–(43), the Clausius–Duhem inequality for the fictitious undamaged continuum in the isothermal conditions is given by

$$\tilde{\sigma}_{ij}\dot{\tilde{\varepsilon}}_{ij}-\dot{\tilde{\Psi}}(\tilde{\varepsilon}_{ij},\tilde{q}^r_{ij},\gamma^s_{ij})\geqslant 0 \tag{44}$$

so that

$$\left(\tilde{\sigma}_{ij} - \frac{\partial \tilde{\Psi}}{\partial \tilde{\varepsilon}_{ij}}\right)\dot{\tilde{\varepsilon}}_{ij} - \frac{\partial \tilde{\Psi}}{\partial \tilde{q}^r_{ij}}\dot{\tilde{q}}^r_{ij} - \frac{\partial \tilde{\Psi}}{\partial \gamma^s_{ij}}\dot{\gamma}^s_{ij} \geqslant 0 \tag{45}$$

In the fictitious continuum, $\tilde{\varepsilon}_{ij}$, $\tilde{q}^r_{ij}$, and γ^s_{ij} are the state variables so that they can be independently varied. Although, $\tilde{q}^r_{ij}$ may vary when $\tilde{\varepsilon}_{ij}$ changes, their relation is not one-to-one. Different $\tilde{\varepsilon}_{ij}$ histories may lead to the same $\tilde{q}^r_{ij}$, and a material with different $\tilde{q}^r_{ij}$ may correspond to the same $\tilde{\varepsilon}_{ij}$ momentarily. Thus, it is possible to vary $\tilde{\varepsilon}_{ij}$ so that $\tilde{q}^r_{ij}$ is left unchanged. Therefore, inequality (45) is always satisfied, if

$$\tilde{\sigma}_{ij} = \frac{\partial \tilde{\Psi}}{\partial \tilde{\varepsilon}_{ij}} \tag{46a}$$

$$-\frac{\partial \tilde{\Psi}}{\partial \tilde{q}^r_{ij}}\dot{\tilde{q}}^r_{ij} - \frac{\partial \tilde{\Psi}}{\partial \gamma^s_{ij}}\dot{\gamma}^s_{ij} \geqslant 0 \tag{46b}$$

According to (46a), the effective stress $\tilde{\sigma}_{ij}$ is derivable from the fictitious undamaged free-energy $\tilde{\Psi}$. The inequality (46b) gives the thermodynamic constraints on the laws governing the evolution of the two sets of internal variables, $\tilde{q}^r_{ij}$ and γ^s_{ij}.

It is now possible to derive the explicit relationship for effective strain $\tilde{\varepsilon}_{ij}$. A relation similar to eqn (46a) exists for the actual damaged continuum. When the postulate of free energy equivalence is assumed, this relation is

$$\sigma_{ij} = \frac{\partial \Psi}{\partial \varepsilon_{ij}} = \frac{\partial \tilde{\Psi}}{\partial \varepsilon_{ij}} = \frac{\partial \tilde{\Psi}}{\partial \tilde{\varepsilon}_{kl}}\frac{\partial \tilde{\varepsilon}_{kl}}{\partial \varepsilon_{ij}} + \frac{\partial \tilde{\Psi}}{\partial \tilde{q}^r_{kl}}\frac{\partial \tilde{q}^r_{kl}}{\partial \varepsilon_{ij}} \tag{47}$$

where the effective internal variable $\tilde{q}^r_{ij}$ is assumed to be a function of the actual internal variable q^r_{ij} and damage tensor D_{ij}. Note that, for the actual damaged continuum, independent variables are ε_{ij}, q^r_{ij} and D_{ij}, so that the second term on the right-hand-side of (47) drops out and the equation reduces to

$$\sigma_{ij} = \frac{\partial \tilde{\Psi}}{\partial \tilde{\varepsilon}_{kl}}\frac{\partial \tilde{\varepsilon}_{kl}}{\partial \varepsilon_{ij}} = \tilde{\sigma}_{kl}\frac{\partial \tilde{\varepsilon}_{kl}}{\partial \varepsilon_{ij}} \tag{48}$$

Using (31), (48) further reduces to

$$\frac{\partial \tilde{\varepsilon}_{kl}}{\partial \varepsilon_{ij}} = N_{ijkl} \tag{49}$$

where N_{ijkl} is the inverse of M_{ijkl} and is a function of D_{ij} only, or

$$M_{ijmn}N_{klij} = I_{mnkl} \tag{50}$$

In (50), the fourth-order identity tensor is $I_{ijkl} = \delta_{ik}\delta_{jl}$ and δ_{ij} is Kronecker's delta. Thus, it follows from (49) that the effective strain $\tilde{\varepsilon}_{ij}$ is linearly related to ε_{ij} by

$$\tilde{\varepsilon}_{ij} = N_{klij}\varepsilon_{kl} \quad \text{or } \varepsilon_{ij} = M_{klij}\tilde{\varepsilon}_{kl} \tag{51a}$$

Then, it is assumed that the following relations are valid for the internal variable q^r_{ij}

$$\tilde{q}^r_{ij} = N_{klij}q^r_{kl} \quad \text{or } q^r_{ij} = M_{klij}\tilde{q}^r_{kl} \tag{51b}$$

Constitutive equations at the fictitious undamaged configuration must satisfy the inequality given by eqn (44). By use of (51a) this inequality can be written as

$$\sigma_{ij}\dot{\varepsilon}_{ij}-\dot{\tilde{\Psi}}(\tilde{\varepsilon}_{ij},\tilde{q}^r_{ij},\gamma^s_{ij})+\tilde{\sigma}_{kl}\frac{\partial\tilde{\varepsilon}_{kl}}{\partial D_{ij}}\dot{D}_{ij}\geqslant 0 \tag{52}$$

where

$$\frac{\partial\tilde{\varepsilon}_{ij}}{\partial D_{mn}}=\frac{\partial N_{klij}}{\partial D_{mn}}\varepsilon_{kl} \tag{53}$$

By observing (41), the first two terms of (52) are the same as the left-hand-side of (37) in the isothermal case. During an incremental loading, the fictitious undamaged continuum undergoes a deformation in the matrix as well as an increase in damage. The first two terms of (52) are energy dissipated associated with this process. However, by definition, the state of the fictitious material remains undamaged at the end of each loading increment. The amount of energy dissipated in order to restore the fictitious continuum to the undamaged sate is represented by the last term of inequality (52). For convenience, inequality (52) can be rewritten as

$$\sigma_{ij}\dot{\varepsilon}_{ij}-\dot{\Psi}(\tilde{\varepsilon}_{ij},\tilde{q}^r_{ij},\gamma^s_{ij})-G_{ij}\dot{D}\geqslant 0 \tag{54}$$

where

$$G_{ij}=-\tilde{\sigma}_{kl}\frac{\partial\tilde{\varepsilon}_{kl}}{\partial D_{ij}} \tag{55}$$

Tensor G_{ij} is the thermodynamic force associated with unit damage growth $\dot{D}_{ij}$, and, in this work it is referred to as the 'damage force' for simplicity. This quantity may also be considered as the energy release rate per unit damage advance. Physically, the negative of the damage force, $-G_{ij}$, can be interpreted as the 'restoring force' which restores the fictitious continuum to its undamaged state after experiencing a unit damage growth $\dot{D}_{ij}$. It is seen from (55) that G_{ij} can be expressed in terms of $\tilde{\sigma}_{ij}$, $\tilde{\varepsilon}_{ij}$ and D_{ij}. A further discussion of the damage force can be found in the Appendix.

5. Plasticity and damage

A discussion is now given to characterize the plastic deformation process, the damage process, and the coupling between the two processes. Starting with the actual damaged configuration, where the state variables are ε_{ij}, D_{ij}, q^r_{ij} and γ^s_{ij}, the Clausius–Duhem inequality (37) can be rewritten for isothermal process as

$$\sigma_{ij}\dot{\varepsilon}_{ij}-\frac{\partial\Psi}{\partial\varepsilon_{ij}}\dot{\varepsilon}_{ij}-\frac{\partial\Psi}{\partial q^r_{ij}}\dot{q}^r_{ij}-\frac{\partial\Psi}{\partial D_{ij}}\dot{D}_{ij}-\frac{\partial\Psi}{\partial\gamma^s_{ij}}\dot{\gamma}^s_{ij}\geqslant 0 \tag{56}$$

Replacing Ψ by $\tilde{\Psi}$ and noting that D_{ij} and γ^s_{ij} are independent variables, eqn (56) becomes

$$\sigma_{ij}\dot{\varepsilon}_{ij}-\frac{\partial\tilde{\Psi}}{\partial\tilde{\varepsilon}_{ij}}\frac{\partial\tilde{\varepsilon}_{ij}}{\partial\varepsilon_{kl}}\dot{\varepsilon}_{kl}-\frac{\partial\tilde{\Psi}}{\partial\tilde{q}^{r}_{ij}}\frac{\partial\tilde{q}^{r}_{ij}}{\partial q^{r}_{kl}}\dot{q}^{r}_{kl}-\left(\frac{\partial\tilde{\Psi}}{\partial\tilde{\varepsilon}_{ij}}\frac{\partial\tilde{\varepsilon}_{ij}}{\partial D_{kl}}+\frac{\partial\tilde{\Psi}}{\partial\tilde{q}^{r}_{ij}}\frac{\partial\tilde{q}^{r}_{ij}}{\partial D_{kl}}\right)\dot{D}_{kl}-\frac{\partial\tilde{\Psi}}{\partial\gamma^{s}_{ij}}\dot{\gamma}^{s}_{ij}\geqslant 0 \tag{57}$$

or, after regrouping of terms, it may be shown that

$$\sigma_{ij}\dot{\varepsilon}_{ij}-\dot{\tilde{\Psi}}(\tilde{\varepsilon}_{ij},\tilde{q}^{r}_{ij},\gamma^{s}_{ij})\geqslant 0 \tag{58}$$

Constraint (58) represents the Clausius–Duhem inequality of the actual damaged configuration. However, unlike (37), inequality (58) involves the fictitious free energy $\tilde{\Psi}(\tilde{\varepsilon}_{ij},\tilde{q}^{r}_{ij},\gamma^{s}_{ij})$ which is defined in the fictitious undamaged configuration, where the fictitious material is isotropic. Therefore, $\tilde{\Psi}(\tilde{\varepsilon}_{ij},\tilde{q}^{r}_{ij},\gamma^{s}_{ij})$ involves only material constants that are isotropic tensors.

In an attempt to characterize the plastic deformation and the damage process, one recognizes that the process does not directly influence the mechanisms of plastic deformation; that is, there is no direct coupling between damage and plastic deformation. In general, plasticity is directly related to slips for metals and to other mechanisms for other materials. In all cases, damage influences plastic strains only because the net area of resistance decreases as the damage proceeds. In the present work, damage does not directly influence plastic deformation of the fictitious undamaged continuum, but it does influence the plastic deformation of the actual continuum.

Based on this observation, the fictitious free energy $\tilde{\Psi}(\tilde{\varepsilon}_{ij},\tilde{q}^{r}_{ij},\gamma^{s}_{ij})$ is assumed to consist of two parts, the fictitious plastic potential $\tilde{\Psi}_1(\tilde{\varepsilon}_{ij},\tilde{q}^{r}_{ij})$ and the damage potential $\tilde{\Psi}_2(D_{ij},\gamma^{s}_{ij})$, i.e.,

$$\tilde{\Psi}(\tilde{\varepsilon}_{ij},\tilde{q}^{r}_{ij},\gamma^{s}_{ij})\equiv\tilde{\Psi}_1(\tilde{\varepsilon}_{ij},\tilde{q}^{r}_{ij})+\tilde{\Psi}_2(D_{ij},\gamma^{s}_{ij}) \tag{59}$$

where potential $\tilde{\Psi}_1(\tilde{\varepsilon}_{ij},q^{s}_{ij})$ characterizes the plastic process of the fictitious undamaged continuum while potential $\tilde{\Psi}_2(D_{ij},\gamma^{s}_{ij})$ describes the damage process.

During deformation, the microcracks and/or microvoids will extend, grow and nucleate, resulting in progressive material deterioration. This damage deterioration is not arbitrary and it must obey thermodynamic constraints to be established. The damage potential $\tilde{\Psi}_2(D_{ij},\gamma^{s}_{ij})$ is used to provide the equation of damage evolution and its necessary constraints.

The state of microdefects is represented by the set of internal variables γ^{s}_{ij}. The change in microdefects together with the loading conditions bring about a decrease in load-resisting area, which is represented at the macroscopic level by damage tensor D_{ij}. The effect is carried over to the deformation process, elastic or plastic, through the effective variables, $\tilde{\sigma}_{ij}$, $\tilde{\varepsilon}_{j}$ and $\tilde{q}^{r}_{ij}$. Hence, an indirect coupling occurs between the plasticity and damage in the actual damaged continuum. Using (59), inequality (58) is written as

$$\sigma_{ij}\dot{\varepsilon}_{ij}-\dot{\tilde{\Psi}}_1(\tilde{\varepsilon}_{ij},\tilde{q}^{r}_{ij})-\dot{\tilde{\Psi}}_2(D_{ij},\gamma^{s}_{ij})\geqslant 0 \tag{60}$$

6. The constitutive equations and constraints

Within an infinitesimal strain theory, the stress rate is usually represented by the material rate. In CDM, it is important, however, to consider the rotation of the principal directions of damage during the deformation process. The principal directions of damage do not generally coincide with the principal stress, when nonproportional loading takes place or when the material has suffered a prior damage. To satisfy the requirements of reference frame indifference, the rate of change of damage measure D_{ij} and the internal variables γ^s_{ij} are expressed by the corotational derivatives

$$\overset{\nabla}{D}_{ij} = \dot{D}_{ij} - \omega_{ik} D_{kj} + D_{ik}\omega_{kj} \tag{61a}$$

$$\overset{\nabla}{\gamma}{}^s_{ij} = \dot{\gamma}^s_{ij} - \omega_{ik}\gamma^s_{kj} + \gamma^s_{ik}\omega_{kj} \tag{61b}$$

where ∇ denotes the corotational differentiation and ω_{ij} represents the spin of the principal damage coordinate frame with respect to the fixed reference coordinate frame. Thus, it follows from (60) that

$$\left(\sigma_{ij} - \frac{\partial \tilde{\Psi}_1}{\partial \tilde{\varepsilon}_{kl}} \frac{\partial \tilde{\varepsilon}_{kl}}{\partial \varepsilon_{ij}}\right)\dot{\varepsilon}_{ij} - \frac{\partial \tilde{\Psi}_1}{\partial \tilde{q}^r_{kl}} \frac{\partial \tilde{q}^r_{kl}}{\partial q^r_{ij}} \dot{q}^r_{ij} - \left(\frac{\partial \tilde{\Psi}_1}{\partial \tilde{\varepsilon}_{kl}} \frac{\partial \tilde{\varepsilon}_{kl}}{\partial D_{ij}} + \frac{\partial \tilde{\Psi}_1}{\partial \tilde{q}^r_{ij}} \frac{\partial \tilde{q}^r_{kl}}{\partial D_{ij}} + \frac{\partial \tilde{\Psi}_2}{\partial D_{ij}}\right)\overset{\nabla}{D}_{ij} - \frac{\partial \tilde{\Psi}_2}{\partial \gamma^s_{ij}} \overset{\nabla}{\gamma}{}^s_{ij} \geqslant 0 \tag{62}$$

Since ε_{ij}, D_{ij}, q^r_{kl} and γ^s_{ij} are independent state variables, fixing these values also fix the values of σ_{ij}, $\tilde{\Psi}_1$, and $\tilde{\Psi}_2$ because they are state functions. For inequality (62) not to be violated for any arbitrary choice of $\dot{\varepsilon}_{ij}$ while keeping D_{ij}, q^r_{ij}, and γ^s_{ij} unchanged, the following conditions must hold

$$\sigma_{ij} = \frac{\partial \tilde{\Psi}_1}{\partial \tilde{\varepsilon}_{kl}} \frac{\partial \tilde{\varepsilon}_{kl}}{\partial \varepsilon_{ij}} \tag{63a}$$

$$-\frac{\partial \tilde{\Psi}_1}{\partial \tilde{q}^r_{kl}} \frac{\partial \tilde{q}^r_{kl}}{\partial q^r_{ij}} \dot{q}^r_{ij} - \left(\frac{\partial \tilde{\Psi}_1}{\partial \tilde{\varepsilon}_{kl}} \frac{\partial \tilde{\varepsilon}_{kl}}{\partial D_{ij}} + \frac{\partial \tilde{\Psi}_1}{\partial \tilde{q}^r_{kl}} \frac{\partial \tilde{q}^r_{kl}}{\partial D_{ij}} + \frac{\partial \tilde{\Psi}_2}{\partial D_{ij}}\right)\overset{\nabla}{D}_{ij} - \frac{\partial \dot{\tilde{\Psi}}_2}{\partial \gamma^s_{ij}} \overset{\nabla}{\gamma}{}^s_{ij} \geqslant 0 \tag{63b}$$

Using (49) and noting that N_{ijkl} is the inverse of M_{ijkl}, (63a) reduces to

$$\tilde{\sigma}_{ij} = \frac{\partial \tilde{\Psi}_1}{\partial \tilde{\varepsilon}_{ij}} \tag{64}$$

Thus, the effective stress is derivable from potential $\tilde{\Psi}_1(\tilde{\varepsilon}_{ij}, \tilde{q}^r_{ij})$. Also, (63b) can be rearranged to yield

$$-\frac{\partial \tilde{\Psi}_1}{\partial \tilde{q}^r_{kl}} \overset{\nabla}{\tilde{q}}{}^r_{kl} + \left(G_{ij} - \frac{\partial \tilde{\Psi}_2}{\partial D_{ij}}\right)\overset{\nabla}{D}_{ij} - \frac{\partial \tilde{\Psi}_2}{\partial \gamma^s_{ij}} \overset{\nabla}{\gamma}{}^s_{ij} \geqslant 0 \tag{65}$$

where

$$G_{ij} = -\frac{\partial \tilde{\Psi}_1}{\partial \tilde{\varepsilon}_{kl}} \frac{\partial \tilde{\varepsilon}_{kl}}{\partial D_{ij}} = -\tilde{\sigma}_{kl} \frac{\partial \tilde{\varepsilon}_{kl}}{\partial D_{ij}} \tag{66}$$

In the derivation, eqns (64) and (55) were used. Note that the first term of (63b) and the second term in the bracket of the same inequality are combined to form the first term of (65).

From the assumption that damage does not directly influence the state of plasticity of the fictitious continuum, i.e., damage affects the deformation only through D_{ij}, the constraint (65) can be replaced by the following stronger conditions

$$-\frac{\partial \tilde{\Psi}_1}{\partial \tilde{q}^r_{ij}} \overset{\nabla}{\tilde{q}}{}^r_{ij} \geqslant 0 \tag{67a}$$

$$\left(G_{ij}-\frac{\partial\tilde{\Psi}_2}{\partial D_{ij}}\right)\overset{\nabla}{D}_{ij}-\frac{\partial\tilde{\Psi}_2}{\partial\gamma^s_{ij}}\overset{\nabla}{\gamma}{}^s_{ij}\geqslant 0 \tag{67b}$$

The conditions apply, respectively, to the fictitious plastic deformation process and the damage process. Furthermore, if the values of ε_{ij}, D_{ij}, q^r_{ij}, and γ^s_{ij} are fixed, then the values of G_{ij} and $\tilde{\Psi}_2$ are also fixed, since G_{ij} is a function of state variables as defined by eqn (66), and $\tilde{\Psi}_2$ is a state function. For inequality (67b) not to be violated for an arbitrary choice of $\overset{\nabla}{D}_{ij}$, the following conditions must hold

$$G_{ij}=\frac{\partial\tilde{\Psi}_2}{\partial D_{ij}} \tag{68}$$

$$-\frac{\partial\tilde{\Psi}_2}{\partial\gamma^s_{ij}}\overset{\nabla}{\gamma}{}^s_{ij}\geqslant 0 \tag{69}$$

For a more detailed investigation of a CDM model, the damage tensor D_{ij} can be divided into two parts, i.e.,

$$D_{ij}=D^{\mathrm{r}}_{ij}+D^{\mathrm{n}}_{ij} \tag{70}$$

The recoverable part D^{r}_{ij} is due to area reduction associated with the growth of microdefects that can be recovered during unloading. The non-recoverable part D^{n}_{ij} involves the reduction of area due to the extension of existing microcracks and/or the nucleation of microdefects.

In summary, the constitutive equations for an isothermal damaged continuum are given by the following sets of equations and constraints

$$\tilde{\sigma}_{ij}=\frac{\partial\tilde{\Psi}_1}{\partial\tilde{\varepsilon}_{ij}}\quad\text{with}\quad -\frac{\partial\tilde{\Psi}_1}{\partial\tilde{q}^r_{ij}}\overset{\nabla}{\tilde{q}}{}^r_{ij}\geqslant 0 \tag{71}$$

$$G_{ij}=\frac{\partial\tilde{\Psi}_2}{\partial D_{ij}}\quad\text{with}\quad -\frac{\partial\tilde{\Psi}_2}{\partial\gamma^s_{ij}}\overset{\nabla}{\gamma}{}^s_{ij}\geqslant 0 \tag{72}$$

and

$$G_{ij}=-\frac{\partial\tilde{\Psi}_1}{\partial\tilde{\varepsilon}_{kl}}\frac{\partial\tilde{\varepsilon}_{kl}}{\partial D_{ij}} \tag{73}$$

The equation of (71) characterizes the deformation of the fictitious undamaged continuum, and the inequality of (71) constrains the evolution of plastic internal variables. The set of equation and constraint (72) provides a relationship between the damage force G_{ij} and the damage measure D_{ij}. It also provides a constraint on the evolution of damage internal variables. Finally, the coupling between the deformation process and damage process is provided by eqn (73). This is further explained in the subsequent paragraph.

Consider a fictitious, undamaged, element subjected to loading increment $\mathrm{d}\tilde{P}_i$. During loading, there are various forms of dissipation of energy associated with plasticity and damage processes. In particular, the rate of energy dissipation (caused by the damage force) due to a unit damage growth $\overset{\nabla}{D}_{ij}$ with respect to the fictitious element is $(\partial\tilde{\Psi}_2/\partial D_{ij})\overset{\nabla}{D}_{ij}$. At the end of the loading period, it is required that the fictitious element returns to its undamaged state before the next loading can be applied. The restoring energy associated with this transformation is given by $-G_{ij}\overset{\nabla}{D}_{ij}$, where $-G_{ij}$ is the restoring force. Because the damage needed to be restored at the end of a loading period is equal to the negative of the damage growth during loading, the force associated with the two processes must be equal in magnitude. Therefore, eqn (73) can be viewed as a constraint that must be satisfied for the fictitious deformation and damage process to occur simultaneously within the same continuum. In fact, this is a required constraint which arrives naturally from thermodynamic consideration.

Equations and constraints in (71)–(73) provide a framework for theories of CDM. Explicit constitutive equations may be obtained if functions for $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ are specified. Explicit evolution equations, which satisfy the inequalities of (71) and (72), for internal variables q^r_{ij} and γ^s_{ij} should also be given. Different theories may be proposed based on this constitutive framework. One such theory has been formulated by Wu and Nanakorn (1998) by use of concept of endochronic plasticity. In that paper, the model has been applied to a one-dimensional case which describes uniaxial monotonic compression and tension of a concrete specimen. It successfully describes the strain-softening behavior after the peak load. In addition, the model has been applied to the description of deformation behavior for cyclically loaded concrete and mortar specimens. Satisfactory results have been obtained.

It is remarked that the internal variables are not observable. Using the evolution equations for these variables, these variables do not necessarily appear in the final form of the constitutive equations. Depending on the functional forms for $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ and the explicit forms for the evolution equations for q^r_{ij}, and γ^s_{ij}, a set of macroscopic parameters may be used for the model. These parameters may then be determined from experiments.

7. A brief summary of authors' endochronic CDM

CDM models based on endochronic theory of plasticity have been previously proposed. The model of Valanis (1990) is for brittle materials, while the models of Niu (1989) and Chow and Chen (1992) are for ductile materials. The model of Niu (1989) is limited to isotropic damage due to the scalar representation of damage; the model of Chow and Chen (1992) is an anisotropic damage model. Wu and Nanakorn's model (1998) is applicable to ductile materials with anisotropic damage. The equations of the Wu–Nanakorn model are summarized in this section for later reference. This model is different than that of Chow and Chen (1992), which used neither the damage internal variables γ^s_{ij} nor the concept of damage restoring force G_{ij}. Instead, Chow and Chen (1992) use D_{ij} as an internal variable and express the damage evolution equations in terms of the potential of damage dissipation and elastic strain energy release rate Y_{ij}.

In the Wu–Nanakorn model, the governing equations and constraints are given by (71)–(73). In this section, explicit forms of equations are derived by assuming the following quadratic forms for $\tilde{\Psi}_1(\tilde{\varepsilon}_{ij}, \tilde{q}^r_{ij})$ and $\tilde{\Psi}_2(D_{ij}, \gamma^s_{ij})$:

$$\tilde{\Psi}_1(\tilde{\varepsilon}_{ij}, \tilde{q}^r_{ij}) = \frac{1}{2}\sum_r A_{ijkl}(\tilde{\varepsilon}_{ij} - \tilde{q}^r_{ij})(\tilde{\varepsilon}_{kl} - \tilde{q}^r_{kl}) \tag{74}$$

$$\tilde{\Psi}_2(D_{ij}, \gamma^s_{ij}) = \frac{1}{2}\sum_r H_{ijkl}(D_{ij} - \gamma^s_{ij})(D_{kl} - \gamma^s_{ij}) \tag{75}$$

where A_{ijkl} and H_{ijkl} are positive semi-definite fourth-order isotropic tensors. The free energies in (74) and (75) are defined on the fictitious undamaged material, which is isotropic. Represent now any fourth order isotropic tensor W_{ijkl} by

$$W_{ijkl} = W_1 \delta_{ij}\delta_{kl} + W_2 \delta_{ik}\delta_{jl}, \quad \text{with } W_0 = 3\left(W_1 + \frac{W_2}{3}\right) \tag{76}$$

where W_1 and W_2 are constants, and symmetry of W_{ijkl} with respect to k and l is assumed. Also, the variables may be decomposed into the deviatoric and hydrostatic parts as

$$\tilde{\sigma}_{ij} = \tilde{s}_{ij} + \tfrac{1}{3}\delta_{ij}\tilde{\sigma}_{kk} \quad \tilde{\varepsilon}_{ij} = \tilde{e}_{ij} + \tfrac{1}{3}\delta_{ij}\tilde{\varepsilon}_{kk} \quad \tilde{q}^r_{ij} = \tilde{p}^r_{ij} + \tfrac{1}{3}\delta_{ij}\tilde{q}^r_{kk} \tag{77a}$$

and

$$G_{ij} = g_{ij} + \tfrac{1}{3}\delta_{ij}G_{kk} \quad D_{ij} = d_{ij} + \tfrac{1}{3}\delta_{ij}D_{kk} \quad \gamma^s_{ij} = r^s_{ij} + \tfrac{1}{3}\delta_{ij}\gamma^s_{kk} \tag{77b}$$

Using these notations, the explicit form of constitutive equations for damaged materials may be derived and are presented in the remaining part of this section.

7.1. Equations of deformation

The deformation behavior is characterized by the effective stress–effective strain relationship given in (71). Using (74), this equation reduces to the following two equations by separating hydrostatic and deviatoric components:

$$\tilde{\sigma}_{kk} = \sum_r A^r_0(\tilde{\varepsilon}_{kk} - \tilde{q}^r_{kk}) \tag{78}$$

$$\tilde{s}_{ij} = \sum_r A^r_2(\tilde{e}_{ij} - \tilde{p}^r_{ij}) \tag{79}$$

where A^r_0 and A^r_2 are defined by (76) with W replaced by A.

Within a linear assumption, the evolution equations for the hydrostatic and deviatoric parts of $\tilde{q}^r_{ij}$ are given in the following form

$$L^r_0{}^{\nabla}\left(\frac{\mathrm{d}\tilde{q}^r_{kk}}{\mathrm{d}\tilde{z}_{\mathrm{H}}}\right) - \tilde{\sigma}_{kk} = 0 \quad \text{and} \quad L^r_2{}^{\nabla}\left(\frac{\mathrm{d}\tilde{p}^r_{ij}}{\mathrm{d}\tilde{z}_{\mathrm{D}}}\right) - \tilde{s}_{ij} = 0 \tag{80}$$

where ${}^{\nabla}()$ indicates that the differentiation operator in brackets is corotational; L^r_0 and L^r_2 are constants; and $\mathrm{d}\tilde{z}$ defines the intrinsic time with respect to the fictitious deformation. The intrinsic time is a monotinically increasing parameter that is used to register the history of deformation in an endochronic theory. The intrinsic time is divided into the hydrostatic and deviatoric parts.

An hydrostatic intrinsic time measure ζ_{H} is defined to register the hydrostatic deformation. It is scaled by the intrinsic time scale z_{H} so that it can properly describe strain-hardening. They are related by

$$\mathrm{d}\zeta_{\mathrm{H}} = \left|\mathrm{d}\tilde{\varepsilon}_{kk} - k_1 \frac{\mathrm{d}\tilde{\sigma}_{kk}}{3K_0}\right| \quad \text{with} \quad \frac{\mathrm{d}\zeta_{\mathrm{H}}}{\mathrm{d}z_{\mathrm{H}}} = h(\zeta_{\mathrm{H}}) > 0 \tag{81}$$

where $0 \leqslant k_1 \leqslant 1$ and K_0 is the Bulk Modulus.

The deviatoric intrinsic time ζ_{D} is defined based on an effective strain-like tensor $\tilde{Q}_{ij}$ which is given by

$$^{\nabla}\mathrm{d}\tilde{Q}_{ij} = {}^{\nabla}\mathrm{d}\tilde{e}_{ij} - k_2 \frac{^{\nabla}\mathrm{d}\tilde{s}_{ij}}{2\mu_0} \tag{82}$$

where $0 \leqslant k_2 \leqslant 1$ and μ_0 is the shear modulus. The operator $^{\nabla}\mathrm{d}$ denotes the corotational increment and is defined on a second-order tensor a_{ij} with respect to the intrinsic time z as

$$^{\nabla}\mathrm{d}a_{ij} = {}^{\mathrm{t}}\mathrm{d}a_{ij} - \omega_{ik}a_{kj}\,\mathrm{d}z + a_{ik}\omega_{kj}\,\mathrm{d}z \tag{83}$$

where $^{\mathrm{t}}\mathrm{d}$ denotes the increment based on material rate and ω_{ij} is the spin tensor.

The deviatoric intrinsic time is defined and scaled as follows:

$$\mathrm{d}\zeta_{\mathrm{D}}^2 = {}^{\nabla}\mathrm{d}\tilde{Q}_{ij}{}^{\nabla}\mathrm{d}\tilde{Q}_{ij} \quad \text{with} \frac{\mathrm{d}\zeta_{\mathrm{D}}}{\mathrm{d}z_{\mathrm{D}}} = f(\zeta_{\mathrm{D}}) > 0 \tag{84}$$

In (81) and (84), $h(\zeta_{\mathrm{H}})$ and $f(\zeta_{\mathrm{D}})$ are the strain hardening functions corresponding to the hydrostatic and deviatoric deformation, respectively.

7.2. Equations of damage

Equations (72) and (75) lead to the following relations for the hydrostatic and deviatoric parts of damage force G_{ij}, respectively,

$$G_{kk} = \sum_s H_0^s (D_{kk} - \gamma_{kk}^s) \tag{85}$$

$$g_{ij} = \sum_s H_2^s (d_{ij} - r_{ij}^s) \tag{86}$$

where H_0^s and H_2^s are constants. In the hydrostatic damage, microdefects expand and/or contract such that the overall symmetric properties of the material, i.e., all planes of symmetry, are retained. In the deviatoric damage, changes in orientation of microcracks and/or microvoids result in changes of overall symmetry properties and, thus, induce the anisotropic behavior of the material.

Using the inequality of (72), the linear evolution equations of γ_{ij}^s can be further separated into hydrostatic and deviatoric parts as

$$J_0^{s\,\nabla}\left(\frac{\mathrm{d}\gamma_{kk}^s}{\mathrm{d}z_{\mathrm{H}}^{\mathrm{d}}}\right) - G_{kk} = 0 \quad \text{and} \quad J_2^{s\,\nabla}\left(\frac{\mathrm{d}r_{ij}^s}{\mathrm{d}z_{\mathrm{D}}^{\mathrm{d}}}\right) - g_{ij} = 0 \tag{87}$$

where J_0^s and J_2^s are constants; $\mathrm{d}z_{\mathrm{H}}^{\mathrm{d}}$ and $\mathrm{d}z_{\mathrm{H}}^{\mathrm{d}}$ are the damage intrinsic time increment corresponding to the hydrostatic and deviatoric damage, respectively. The hydrostatic damage intrinsic time and its time scale are defined, respectively, by

$$d\zeta_{\mathrm{H}}^{\mathrm{d}} = \left| dD_{kk} - k_3 \frac{dG_{kk}}{3B_0} \right| \quad \text{with} \quad \frac{d\zeta_{\mathrm{H}}^{\mathrm{d}}}{dz_{\mathrm{H}}^{\mathrm{d}}} = h^{\mathrm{d}}(\zeta_{\mathrm{H}}^{\mathrm{d}}) > 0 \tag{88}$$

where $0 \leqslant k_3 \leqslant 1$ and B_0 is a material constant. Similarly, the deviatoric damage intrinsic time and its time scale are defined by

$$(d\zeta_{\mathrm{D}}^{\mathrm{d}})^2 = {}^{\nabla}dR_{ij}{}^{\nabla}dR_{ij} \quad \text{with} \quad \frac{d\zeta_{\mathrm{D}}^{\mathrm{d}}}{dz_{\mathrm{D}}^{\mathrm{d}}} = f^{\mathrm{d}}(\zeta_{\mathrm{D}}^{\mathrm{d}}) > 0 \tag{89a}$$

where the damage-like tensor ${}^{\nabla}dR_{ij}$ is defined by

$${}^{\nabla}dR_{ij} = {}^{\nabla}dd_{ij} - k_4 \frac{{}^{\nabla}dg_{ij}}{2M_0} \tag{89b}$$

with $0 \leqslant k_4 \leqslant 1$ and M_0 is a material constant. The role played by material constants B_0 and M_0 in the G_{ij} vs D_{ij} relationship is similar to that played by the bulk modulus K_0 and shear modulus μ_0 in the stress–strain relationship. The functions $h^{\mathrm{d}}(\zeta_{\mathrm{H}}^{\mathrm{d}})$ and $f^{\mathrm{d}}(\zeta_{\mathrm{D}}^{\mathrm{d}})$ describe the material damage resisting (hardening) behavior, which increases the damage threshold. These functions are similar to the hardening functions $h(\zeta_{\mathrm{H}})$ and $f(\zeta_{\mathrm{D}})$ of plastic deformation, but with a different physical meaning.

The role played by (88) and (89) in damage is analogous to that played by (81) and (84) in the stress–strain space for the limit case of $k_i \to 1$ ($i = 1, 2, 3, 4$). The relations (88a) and (89b) can be interpreted as the non-recoverable hydrostatic and deviatoric parts of the damage tensor increment dD_{ij}, respectively. Note that ζ^{d} is defined in terms of the non-recoverable damage D_{ij}^{n} rather than the effective plastic strain $\tilde{\varepsilon}_{ij}^{\mathrm{p}}$, as in the theories of Niu (1989) and Chow and Chen (1992). This new damage intrinsic time enables the present theory to describe the behavior of damage in brittle materials, where damage occurs within the elastic range, as well as in ductile materials.

7.3. Coupling between deformation and damage

The coupling between the deformation and damage process is achieved through damage force G_{ij} given by (73). Using the quadratic form of $\tilde{\Psi}_1(\tilde{\varepsilon}_{ij}, \tilde{q}_{ij}^r)$ given in (74), (73) reduces to

$$G_{ij} = -\sum_r A_{klmn}^r (\tilde{\varepsilon}_{mn} - \tilde{q}_{mn}^r)^{\nabla}\left(\frac{\partial \tilde{\varepsilon}_{kl}}{\partial D_{ij}}\right) = -\sum_r A_{klmn}^r (\tilde{\varepsilon}_{mn} - \tilde{q}_{mn}^r)\varepsilon_{pq}{}^{\nabla}\left(\frac{dN_{pqkl}}{dD_{ij}}\right) \tag{90}$$

Equation (90) relates the damage force G_{ij} to the effective strain $\tilde{\varepsilon}_{ij}$ and effective internal variables $\tilde{q}_{ij}^r$, both of which are responsible for the deformation process of the fictitious continuum. On the other hand, the damage force is related to the damage potential $\tilde{\Psi}_2(D_{ij}, \gamma_{ij}^s)$ by eqn (72), which, by use of (75), reduces to

$$G_{ij} = \frac{\partial \tilde{\Psi}_2}{\partial D_{ij}} = \sum_r H_{ijkl}^s (D_{kl} - \gamma_{kl}^s) \tag{91}$$

This equation relates damage force G_{ij} to the damage tensor D_{ij} and internal variables γ_{ij}^s. The interpretations of (90) and (91) are as follows: consider a fictitious, undamaged, material element subjected to a loading increment. During loading, different forms of energy associated with plasticity and damage process are dissipated. In particular, the rate of energy dissipation due to damage growth $\overset{\nabla}{D}_{ij}$ in the fictitious element is $(\partial\Psi_2/\partial D_{ij})\overset{\nabla}{D}_{ij}$. At the end of the loading period, it is required that the fictitious element returns to its undamaged state before the next loading can be applied. The restoring energy associated with this process is $-G_{ij}\overset{\nabla}{D}_{ij}$, where $-G_{ij}$ is the restoring force and the negative of the restoring force, G_{ij}, is given by (90). Because the damage to be restored at the end of the loading period is equal to the negative of the damage growth during loading, the force associated with the two processes must be equal in magnitude. Therefore, (91) with G_{ij} defined by (90), can be viewed as a constraint that must be satisfied for the deformation and damage processes to occur simultaneously within the fictitious continuum. In fact, this is a required constraint which arises naturally from the thermodynamic consideration.

8. Application

In this section, the model of the previous section is applied to investigate the problem of a cylindrical concrete specimen subjected to uniaxial compression in the x_3-direction. For such a problem, the state of stress and strain is given by

$$\sigma_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \quad \varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \tag{92}$$

where ε_{33} is prescribed for a strain control test. These are the principal stress and strain components and the directions of principal damage coincide with those of the principal stress and strain, if the specimen is initially isotropic and is subjected to proportional loading. In this case, the corotational rate reduces to the material rate. The increments of effective stress and effective strain are

$$\mathrm{d}\tilde{\sigma}_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{d}\tilde{\sigma}_{33} \end{bmatrix} \quad \mathrm{d}\tilde{\varepsilon}_{ij} = \begin{bmatrix} \mathrm{d}\tilde{\varepsilon}_{11} & 0 & 0 \\ 0 & \mathrm{d}\tilde{\varepsilon}_{22} & 0 \\ 0 & 0 & \mathrm{d}\tilde{\varepsilon}_{33} \end{bmatrix} \tag{93}$$

Their deviatoric parts are

$$\mathrm{d}\tilde{s}_{ij} = \frac{1}{3}\begin{bmatrix} -\mathrm{d}\tilde{\sigma}_{33} & 0 & \\ 0 & -\mathrm{d}\tilde{\sigma}_{33} & 0 \\ 0 & 0 & 2\mathrm{d}\tilde{\sigma}_{33} \end{bmatrix} \tag{94a}$$

$$\mathrm{d}\tilde{e}_{ij} = \frac{1}{3}\begin{bmatrix} 2\mathrm{d}\tilde{\varepsilon}_{11}-\mathrm{d}\tilde{\varepsilon}_{22}-\mathrm{d}\tilde{\varepsilon}_{33} & 0 & 0 \\ 0 & -\mathrm{d}\tilde{\varepsilon}_{11}+2\mathrm{d}\tilde{\varepsilon}_{22}-\mathrm{d}\tilde{\varepsilon}_{33} & 0 \\ 0 & 0 & -\mathrm{d}\tilde{\varepsilon}_{11}-\mathrm{d}\tilde{\varepsilon}_{22}+2\mathrm{d}\tilde{\varepsilon}_{33} \end{bmatrix} \tag{94b}$$

The damage effect tensor M_{ijkl} is selected according to Bitten's definition of effective stress. Using (16) and (25), it is expressed in the matrix form as

$$[M]_{ij} = \begin{bmatrix} \dfrac{1}{(1-D_1)^2} & 0 & 0 \\ 0 & \dfrac{1}{(1-D_2)^2} & 0 \\ 0 & 0 & \dfrac{1}{(1-D_3)^2} \end{bmatrix} \tag{95a}$$

The inverse of this matrix is

$$[N]_{ij} = \begin{bmatrix} (1-D_1)^2 & 0 & 0 \\ 0 & (1-D_2)^2 & 0 \\ 0 & 0 & (1-D_3)^2 \end{bmatrix} \tag{95b}$$

where D_1, D_2, and D_3 are the principal damage in the x_1-, x_2- and x_3-directions, respectively.

In this example, for the sake of simplicity, only one internal variable each for $\tilde{q}_{ij}$ and γ_{ij} is used. The use of only one internal variable was shown in previous applications of endochronic plasticity to be capable of capturing the main features of stress–strain responses in a plastically deformed continuum. The hydrostatic behavior of the fictitious deformation is now considered. Combining (78), (80a) and (81a), the following equation is obtained

$$\frac{\mathrm{d}\tilde{\sigma}_{kk}}{A_0} \pm k_1 X \frac{\mathrm{d}\tilde{\sigma}_{kk}}{3K_0} = (1 \pm X)\,\mathrm{d}\tilde{\varepsilon}_{kk} \tag{96}$$

where

$$X = \frac{\tilde{\sigma}_{kk}}{L_0 h(\zeta_{\mathrm{H}})} \tag{97}$$

and A_0 and L_0 are constants. The minus $(-)$ and plus $(+)$ signs, in (96), correspond to tension and compression, respectively. The material constant A_0 may be identified with the bulk modulus $3K_0$ by considering the fictitious undamaged material as its initial loading state, where $\tilde{\sigma}_{kk} = \tilde{q}_{kk} = 0$. Using the effective stress and effective strain of (93), (96) becomes

$$\mathrm{d}\tilde{\sigma}_{33} = 3K_0 F(k_1, X)(\mathrm{d}\tilde{\varepsilon}_{11} + \mathrm{d}\tilde{\varepsilon}_{22} + \mathrm{d}\tilde{\varepsilon}_{33}) \tag{98}$$

where

$$F(k_1, X) = \frac{1 \pm X}{1 \pm k_1 X} \quad \text{and} \quad X = \frac{\tilde{\sigma}_{33}}{L_0 h(\zeta_{\mathrm{H}})} \tag{99}$$

The deviatoric response of the fictitious undamaged material, from (79), (80b) and (94), is described by

$$-\mathrm{d}\tilde{\sigma}_{33} = A_2(2\,\mathrm{d}\tilde{\varepsilon}_{11} - \mathrm{d}\tilde{\varepsilon}_{22} - \mathrm{d}\tilde{\varepsilon}_{33}) + \frac{A_2 \tilde{\sigma}_{33}}{L_2}\,\mathrm{d}z_{\mathrm{D}} \tag{100a}$$

$$-\mathrm{d}\tilde{\sigma}_{33} = A_2(-\mathrm{d}\tilde{\varepsilon}_{11}+2\mathrm{d}\tilde{\varepsilon}_{22}-\mathrm{d}\tilde{\varepsilon}_{33})+\frac{A_2\tilde{\sigma}_{33}}{L_2}\mathrm{d}z_\mathrm{D} \tag{100b}$$

$$2\mathrm{d}\tilde{\sigma}_{33} = A_2(-\mathrm{d}\tilde{\varepsilon}_{11}-\mathrm{d}\tilde{\varepsilon}_{22}+2\,\mathrm{d}\tilde{\varepsilon}_{33})-\frac{A_2\tilde{\sigma}_{33}}{L_2}\mathrm{d}z_\mathrm{D} \tag{100c}$$

Equation (98) and the two independent equations of (100) can be put in the matrix form as

$$\begin{bmatrix} 1 & -3K_0F(k_1,x) & -3K_0F(k_1,x) \\ 1 & 2A_2 & -A_2 \\ 1 & -A_2 & 2A_2 \end{bmatrix}\begin{Bmatrix} \mathrm{d}\tilde{\sigma}_{33} \\ \mathrm{d}\tilde{\varepsilon}_{11} \\ \mathrm{d}\tilde{\varepsilon}_{22} \end{Bmatrix}+\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}\frac{A_2\tilde{\sigma}_{33}}{L_2}\mathrm{d}z_\mathrm{D} = \begin{Bmatrix} 3K_0F(k_1,x) \\ A_2 \\ A_2 \end{Bmatrix}\mathrm{d}\tilde{\varepsilon}_{33} \tag{101}$$

where $\mathrm{d}z_\mathrm{D}$ is related to the deviatoric components of the incremental effective strain $\mathrm{d}\tilde{\varepsilon}_{ij}$ and its relationship is now discussed. The following expression may be found from (82)

$$\mathrm{d}\tilde{Q}_{ij} = \alpha_{ij}+\beta_{ij}\,\mathrm{d}z_\mathrm{D} \tag{102a}$$

where

$$\alpha_{ij} = (1-k_2)\,\mathrm{d}\tilde{e}_{ij} \quad \text{and} \quad \beta_{ij} = \frac{k_2\tilde{s}_{ij}}{L_2} \tag{102b}$$

By use of (84) and (102a), the relation for $\mathrm{d}z_\mathrm{D}$ is obtained as

$$\alpha_{ij}\alpha_{ij}+2\alpha_{ij}\beta_{ij}\,\mathrm{d}z_\mathrm{D}+(\beta_{ij}\beta_{ij}-f(\zeta_\mathrm{D})^2)\,\mathrm{d}z_\mathrm{D}^2 = 0 \tag{103}$$

The equation of hydrostatic damage is found from (85), (87a) and (88a) as

$$\frac{\mathrm{d}G_{kk}}{H_0}\pm k_3Y\frac{\mathrm{d}G_{kk}}{3B_0} = (1\pm Y)\,\mathrm{d}D_{kk} \tag{104a}$$

where

$$Y = \frac{G_{kk}}{J_0h^\mathrm{d}(\zeta_\mathrm{H}^\mathrm{d})} \tag{104b}$$

and H_0, B_0 and J_0 are material constants. Considering the initial loading state, where $G_{kk} = \gamma_{kk} = 0$, it may be shown that H_0 is the initial slope of the G_{kk} vs D_{kk} curve and that $H_0 = 3B_0$. The minus $(-)$ and plus $(+)$ signs, in (104a), correspond to tension and compression, respectively. Equation (104a) further reduces to

$$\mathrm{d}G_{kk} = H_0F(k_3,Y)\,\mathrm{d}D_{kk} \tag{105}$$

where

$$F(k_3,Y) = \frac{1\pm Y}{1\pm k_3Y} \tag{106}$$

The equation for deviatoric damage response may be obtained from (86) and (87b) as

$$\mathrm{d}g_{ij} = H_2\left(\mathrm{d}d_{ij} + \frac{g_{ij}}{J_2}\mathrm{d}z_{\mathrm{D}}^{\mathrm{d}}\right) \tag{107}$$

where H_2 and J_2 are constants and H_2 may be identified with the initial slope of the deviatoric g_{ij} vs D_{ij} curve. Furthermore, $H_2 = 2M_0$. The damage intrinsic time increment $\mathrm{d}z_{\mathrm{D}}^{\mathrm{d}}$ is related to the deviatoric components of the damage force increment $\mathrm{d}g_{ij}$ and its relationship is now discussed. The following expression may be found from (89b) and (107)

$$\mathrm{d}R_{ij} = \alpha_{ij}^{\mathrm{d}} + \beta_{ij}^{\mathrm{d}}\,\mathrm{d}z_{\mathrm{D}}^{\mathrm{d}} \tag{108a}$$

where

$$\alpha_{ij}^{\mathrm{d}} = (1-k_4)\frac{\mathrm{d}g_{ij}}{H_2} \quad \text{and} \quad \beta_{ij}^{\mathrm{d}} = \frac{g_{ij}}{J_2} \tag{108b}$$

By use of (89a) and (108a), the relation for $\mathrm{d}z_{\mathrm{D}}^{\mathrm{d}}$ is obtained as

$$\alpha_{ij}^{\mathrm{d}}\alpha_{ij}^{\mathrm{d}} + 2\alpha_{ij}^{\mathrm{d}}\beta_{ij}^{\mathrm{d}}\,\mathrm{d}z_{\mathrm{D}}^{\mathrm{d}} + (\beta_{ij}^{\mathrm{d}}\beta_{ij}^{\mathrm{d}} - f^{\mathrm{d}}(\zeta_{\mathrm{D}}^{\mathrm{d}})^2)(\mathrm{d}z_{\mathrm{D}}^{\mathrm{d}})^2 = 0 \tag{109}$$

Using (73) and (93), the damage force G_{ij} reduces to

$$G_{ij} = -\tilde{\sigma}_{kl}\frac{\partial\tilde{\varepsilon}_{kl}}{\partial D_{ij}} = -\tilde{\sigma}_{33}\frac{\partial\tilde{\varepsilon}_{33}}{\partial D_{ij}} \tag{110}$$

Since $\tilde{\varepsilon}_{33} = (1-D_3)^2\varepsilon_{\mathrm{ee}}$, $\tilde{\varepsilon}_{33}$ does not depend on D_1 and D_2. Consequently, the only non-zero component of tensor G_{ij} is

$$G_{33} = \frac{2\tilde{\sigma}_{33}\tilde{\varepsilon}_{33}}{1-D_3} \tag{111a}$$

and its increment is found to be

$$\mathrm{d}G_{33} = \frac{2\tilde{\sigma}_{33}}{1-D_3}\mathrm{d}\tilde{\varepsilon}_{33} + \frac{2\tilde{\varepsilon}_{33}}{1-D_3}\mathrm{d}\tilde{\sigma}_{33} + \frac{2\tilde{\sigma}_{33}\tilde{\varepsilon}_{33}}{(1-D_3)^2}\mathrm{d}D_3 \tag{111b}$$

Thus, the deviatoric part of the increment of the damage force is

$$\mathrm{d}g_{ij} = \frac{1}{3}\begin{bmatrix} -\mathrm{d}G_{33} & 0 & 0 \\ 0 & -\mathrm{d}G_{33} & 0 \\ 0 & 0 & 2\mathrm{d}G_{33} \end{bmatrix} \tag{111c}$$

Furthermore,

$$\mathrm{d}d_{ij} = \frac{1}{3}\begin{bmatrix} 2\mathrm{d}D_1 - \mathrm{d}D_2 - \mathrm{d}D_3 & 0 & 0 \\ 0 & -\mathrm{d}D_1 + 2\mathrm{d}D_2 - \mathrm{d}D_3 & 0 \\ 0 & 0 & -\mathrm{d}D_1 - \mathrm{d}D_2 + 2\mathrm{d}D_3 \end{bmatrix} \tag{112}$$

Equations (111c) and (112) are substituted into (105) and the two independent equations of (107) to obtain the following matrix equation for the damage process:

$$\begin{bmatrix} H_0F(k_3,y) & H_0F(k_3,y) & H_0F(k_3,y) \\ 2H_2 & -H_2 & -H_2 \\ -H_2 & 2H_2 & -H_2 \end{bmatrix} \begin{Bmatrix} \mathrm{d}D_1 \\ \mathrm{d}D_2 \\ \mathrm{d}D_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \\ -1 \end{Bmatrix} \mathrm{d}G_{33} - \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} \frac{H_2 G_{33}}{J_2} \mathrm{d}z_{\mathrm{D}}^{\mathrm{d}} \tag{113}$$

The equations derived in this section are now applied to the problem of uniaxial compression of cylindrical concrete specimen ($f'_c = 73.8$ Mpa and $E = 27.6$ GPa). To determine the material parameters for the model, the analytical stress–strain curves obtained by Fonseka and Krajcinovic (1981) is used. In this case, the directions of the principal damage coincide with those of the principal stress. The procedure of calculation is now described. An increment $\mathrm{d}\tilde{\varepsilon}_{33}$ is first specified. An initial value for $\mathrm{d}z_{\mathrm{D}}$ is assumed and equation (101) solved for $\mathrm{d}\tilde{\sigma}_{33}$, $\mathrm{d}\tilde{\varepsilon}_{11}$ and $\mathrm{d}\tilde{\varepsilon}_{22}$. These values are then used in (103) to solve for $\mathrm{d}z_{\mathrm{D}}$. An iteration procedure is applied to determine $\mathrm{d}z_{\mathrm{D}}$ for the specified $\mathrm{d}\tilde{\varepsilon}_{33}$. An initial value for $\mathrm{d}D_3$ is then assumed, which is used in (111b) to determine $\mathrm{d}G_{33}$. Equation (109) is subsequently used to determine $\mathrm{d}z_{\mathrm{D}}^{\mathrm{d}}$. Thus, $\mathrm{d}D_1$, $\mathrm{d}D_2$, and $\mathrm{d}D_3$ are found from (113). An iteration procedure is also applied on $\mathrm{d}D_3$ to determine its value which corresponds to the specified $\mathrm{d}\tilde{\varepsilon}_{33}$. Knowing $\mathrm{d}\tilde{\varepsilon}_{33}$ and $\mathrm{d}D_3$, $\mathrm{d}\varepsilon_{33}$ can be calculated from the incremental form of

$$\varepsilon_{33} = \frac{\tilde{\varepsilon}_{33}}{(1-D_3)^2} \tag{114}$$

This procedure continues for another specified $\mathrm{d}\tilde{\varepsilon}_{33}$.

Using a trial-and-error (curve fitting) procedure, the following material parameters for the deformation equations have been determined: Poisson's ratio = 0.2, effective hydrostatic yield stress $L_0 = 4.55$ Mpa, effective deviatoric yield stress $L_2 = 18.61$ Mpa, strain hardening parameters $\beta_{\mathrm{H}} = \beta_{\mathrm{d}} = 0$, and $k_1 = k_2 = 0.95$. The material parameters for the damage equations have been found to be: hydrostatic damage modulus $H_0 = 0.78$ MPa, deviatoric damage modulus $H_2 = 1.91$ MPa, hydrostatic damage threshold $J_0 = 0.61$ MPa, deviatoric damage threshold $J_2 = 1.93$ MPa, damage resisting (hardening) parameters $\beta_{\mathrm{H}}^{\mathrm{d}} = \beta_{\mathrm{D}}^{\mathrm{d}} = 0$, and $k_3 = \mathrm{k}_4 = 0.95$.

The computed stress–strain curves are plotted in Fig. 3. There are two curves in this figure. One curve is for the axial strain and the other for the lateral strain. The volumetric strain vs compressive stress is plotted in Fig. 4. The curve shows a change of sign of the volumetric strain as the axial strain increases. The volumetric strain is initially negative and it changes to positive when the axial strain becomes large in magnitude. This phenomenon is typical in concrete and rocks, and it is due to the increase in the lateral-to-axial strain ratio $[-(\varepsilon_{11}/\varepsilon_{33})]$ as the axial strain increases. It is seen in Fig. 5 that this ratio changes from 0.2 to approximately 0.6 as the axial compressive strain increases from zero to 0.005. The results of this model presented in Figs 3 and 4 show good agreement with the computed value obtained by Fonseka and Krajcinovic (1981).

9. Conclusions

The concepts of CDM have been discussed and a constitutive framework of CDM has been developed based on the internal variables approach. The framework involves transforming the actual damaged continuum into an equivalent fictitious undamaged continuum. The effects of damage are accounted for by replacing the actual stress σ_{ij} (gross stress) on the damaged continuum

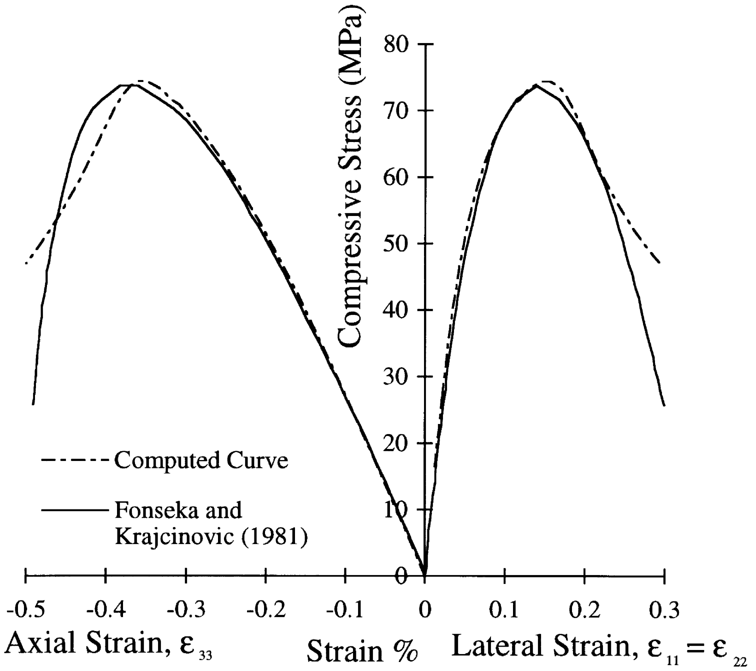

Fig. 3. Stress vs axial and lateral strains in the uniaxial compression of concrete specimen.

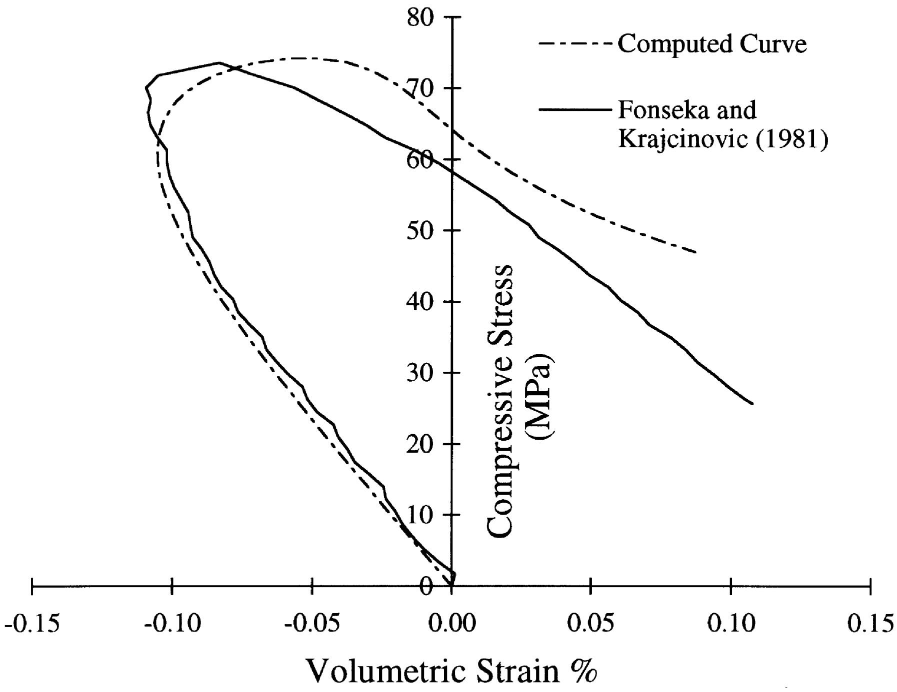

Fig. 4. Stress vs volumetric strain in the uniaxial compression of concrete specimen.

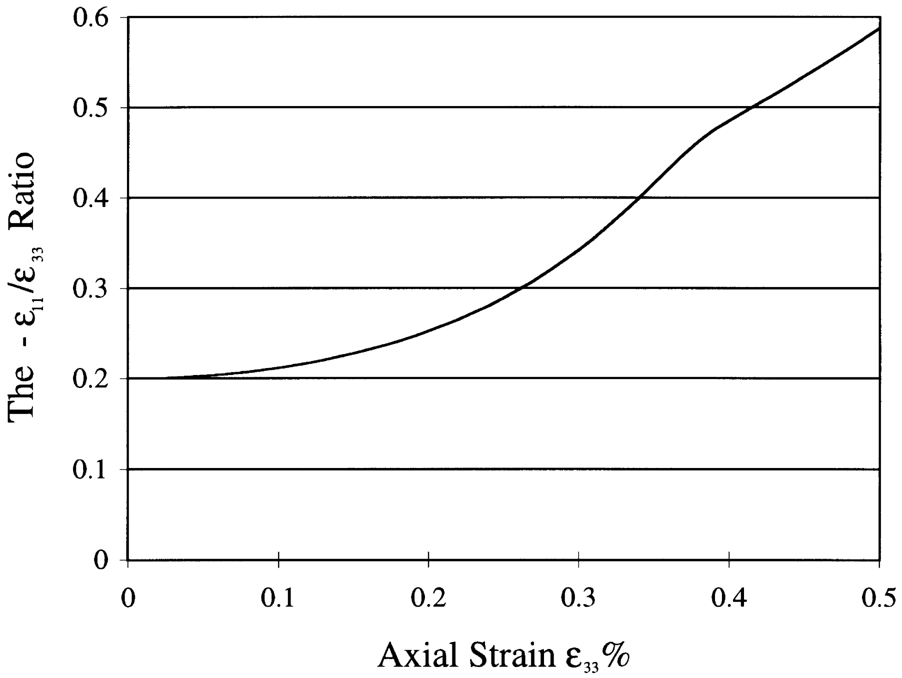

Fig. 5. Lateral to axial strain ratio vs axial compressive strain.

with the symmetric effective stress $\tilde{\sigma}_{ij}$. A distinction has been made between the state of damage and the damage measure D_{ij} and the concept of 'damage force' has been introduced.

Within the proposed constitutive framework, the endochronic concept has been used to derive explicit constitutive equations. Two intrinsic times are used in the formulation. The first intrinsic time ζ is used to describe the plastic deformation history of the fictitious undamaged continuum and the second intrinsic time ζ^{d} is used to depict the damage history. The model is applicable to both brittle and ductile materials with damage.

The following conclusions may be drawn from this study:

(1) The damage tensor D_{ij} may be defined based on a second-order continuity tensor ψ_{ij}.
(2) The damage effect tensor M_{ijkl} defined by Betten (1983) gives rise to an effective stress which has a simple physical interpretation, while other definitions of M_{ijkl} do not have the same significance.
(3) The transformation equation for effective strain, eqn (51a), may be derived based on the free energy equivalence postulate.
(4) In addition to damage tensor D_{ij}, which is a measurable quantity, a set of damage internal variables γ^s_{ij}, which are not measurable, is used in the formulation.
(5) The constitutive equations for an isothermal damaged continuum include two sets of equations and constraints. The first set characterizes the deformation of the fictitious undamaged continuum and constrains the evolution of plastic internal variables q^r_{kl}. The second set provides a relationship between the damage force G_{ij} and the damage measure D_{ij}. It also provides a

constraint on the evolution of damage internal variables γ^s_{ij}. In addition, eqn (73) must be used to complete the constitutive equations. This equation is a constraint that must be satisfied for the fictitious deformation and damage process to occur simultaneously within the same continuum.

(6) The theory does not use the concepts of yield surface or damage surface as its prime requisite although both surfaces may be defined when necessary by setting all $k_i = 1$. Therefore, the constitutive equations of this theory are continuous without discontinuities, which is advantageous in the numerical calculation.

(7) The proposed model has been shown to describe the three-dimensional state of deformation of a cylindrical concrete specimen subjected to uniaxial compression.

The focal point of this paper is to formulate a constitutive framework that is self-consistent. Some well-known concepts have been discussed and it has been pointed out that some concepts are not compatible with others. Only concepts that are compatible to each other are used in the derivation. New concepts such as the distinction between the state of damage and damage measure and the concept to restore the fictitious continuum to its undamaged state after each step of deformation and damage have been introduced in this paper. The paper uses the corotational rate to account for rotation of principal damage directions during deformation.

Appendix

In this study, the damage force G_{ij} is a thermodynamic force associated with a unit damage growth $\overset{\nabla}{D}_{ij}$. This definition of thermodynamic force, eqn (55), is different than the definition that is often defined in CDM. Commonly, the thermodynamic force is known as the damage energy release rate $\underset{\sim}{Y}$ associated with a unit damage growth $\overset{\nabla}{D}_{ij}$ and can be defined in the principal damage coordinate frame, where $\overset{\nabla}{D}_{ij}$ reduces to $\dot{D}_{ij}$, as

$$Y_{ij}\dot{D}_{ij} \geqslant 0 \quad \text{with } Y_{ij} = -\frac{\partial \Psi}{\partial D_{ij}} \tag{A1}$$

where Ψ is the Helmholtz free energy. The concept was first introduced by Chaboche (1977), as an analogy to the energy release rate associated with crack extension in Fracture Mechanics, and it has been adopted by many studies in CDM (Ju, 1989; Woo and Li, 1992; Chow and Chen, 1992). Using the notations of this writing, the damage energy release rate Y_{ij} becomes

$$Y_{ij} = -\frac{\partial \tilde{\Psi}(\tilde{\varepsilon}_{ij}, \tilde{q}^r_{ij}, \gamma^s_{ij})}{\partial D_{ij}} = -\frac{\partial \tilde{\Psi}_1}{\partial \tilde{\varepsilon}_{kl}}\frac{\partial \tilde{\varepsilon}_{kl}}{\partial D_{ij}} - \frac{\partial \tilde{\Psi}_1}{\partial \tilde{q}^r_{kl}}\frac{\partial \tilde{q}^r_{kl}}{\partial D_{ij}} - \frac{\partial \tilde{\Psi}_2}{\partial D_{ij}} \tag{A2}$$

Comparing eqns (66) and (A2), it is clear that the damage force G_{ij} is different than the damage energy release rate Y_{ij} as defined by eqn (A1). However, if the elastic action is the only consideration, the damage force G_{ij} can be compared to the elastic strain energy release rate associated with a unit damage growth defined by Chaboche (1977) and given by

$$Y_{ij} = -\frac{\partial \Psi_e}{\partial D_{ij}} = -\frac{\partial \Psi_e}{\partial \tilde{\varepsilon}^e_{kl}}\frac{\partial \tilde{\varepsilon}^e_{kl}}{\partial D_{ij}} \tag{A3}$$

where $\Psi_e = \frac{1}{2} C_{ijkl}\tilde{\varepsilon}^e_{ij}\tilde{\varepsilon}^e_{kl}$ is the fictitious elastic strain energy, with $\tilde{\varepsilon}^e_{ij}$ representing the effective elastic strain. In this case, the Helmholtz free energy $\tilde{\Psi}$, given by eqn (59), reduces to

$$\tilde{\Psi}(\tilde{\varepsilon}^e_{ij}, \gamma^s_{ij}) = \Psi_1(\tilde{\varepsilon}^e_{ij}) + \Psi_2(D_{ij}, \gamma^s_{ij}) \tag{A4}$$

where the potential Ψ_1 is now identical to the elastic strain energy Ψ_e. Thus, eqn (66) becomes

$$G_{ij} = -\frac{\partial \Psi_1}{\partial \tilde{\varepsilon}^e_{kl}} \frac{\partial \tilde{\varepsilon}^e_{kl}}{\partial D_{ij}} = -\frac{\partial \Psi_e}{\partial \tilde{\varepsilon}^e_{kl}} \frac{\partial \tilde{\varepsilon}^e_{kl}}{\partial D_{ij}} \tag{A5}$$

which is identical to the expression of eqn (A3).

References

Betten, J., 1983. Damage tensors in continuum mechanics. Journal de Mecanique theorique et appliquee 2, 13–32.

Chaboche, J.L., 1977. Sur l'utilisation des variables de'état interne pour la description du comportement viscoplastique et de la rupture par endommagement. Symposium Franco-Polonais de Rhéologie et Mécanique. Cracovie.

Chow, C.L., Chen, X.F., 1992. An anisotropic model of damage mechanics based on endochronic theory of plasticity. Int. J. Fracture 55, 115–130.

Chow, C.L., Lu, T.J., 1989. On evolution laws of anisotropic damage. Engineering Fracture Mechanics 34, 679–701.

Chow, C.L., Lu, T.J., 1992. A comparative study of continuum damage models for crack propagation under gross yielding. Int. J. Fracture 53, 54–75.

Chow, C.L., Wang, J., 1987. An anisotropic theory of elasticity for continuum damage mechanics. Int. J. Fracture 33, 3–16.

Cordebois, J.P., Sidoroff, F., 1979. Damage induced elastic anisotropy. In: Boehler, J.P. (Ed.), Comportement mechanique des solides anisotropes, EUROMECH, Vol. 115. Villard-de-Lens, France. Proc. Coll., Int. du C.N.R.S., pp. 761–774.

Fonseka, G.U., Krajcinovic, D., 1981. The continuous damage theory of brittle materials, Part 2: uniaxial and plane response modes. J. Appl. Mech. 48, 816–824.

Ju, J.W., 1989. On energy based coupled elastoplastic damage theories: constitutive modeling and computational aspects. Int. J. Solids Struct. 25, 803–833.

Kachanov, L.M., 1958. On the time to failure during creep. Izv. AN. SSSR. OTN, pp. 337–360.

Krajcinovic, D., 1985. Continuous damage mechanics revisited: basic concepts and definition. ASME, J. Appl. Mech. 52, 829–834.

Lemaitre, J., 1985. A continuum damage mechanics model for ductile fracture. ASME, J. Engng Mater. Technol. 107, 83–89.

Lemaitre, J., Chaboche, J.L., 1978. Aspects phénoménologique ge la rupture par endommagement. J. Méc. Appl. 2, 167–189.

Lemaitre, J., Chaboche, J.L., 1985. Mécanique des Matériaux Solides. Dunod, Paris. English Edition, Cambridge University Press.

Murakami, S., Ohno, N., 1981. A continuum theory of creep and creep damage. In Ponter, A.R.S., Hayhust, D.P. (Eds.), Creep Structure. Springer, Berlin, pp. 922–944.

Niu, X., 1989. Endochronic plastic constitutive equations coupled with isotropic damage and damage evolution models. Eur. J. Mech., A/Solids 8, 293–308.

Rabotnov, Y.N., 1968. Creep rupture. In Hetenyi, M., Vincenti, H. (Ed.), Proceedings, Applied Mechanics Conference. Standford University, pp. 342–349.

Rabotnov, Y.N., 1969. Creep problems in structural members. North-Holland Series in Applied Math. and Mech., Vol. 7, Chap. 6. North-Holland Publishing Company.

Sidoroff, F., 1981. Description of anisotropic damage application to elasticity. In Hult, J., Lemaitre, J. (Eds.), Physical Non-linearities in Structure Analysis—IUTAM. Springer-Verlag, New York, pp. 237–244.

Stumvoll, M., Swobada, G., 1993. Deformation behavior of ductile solids containing anisotropic damage. J. of Engineering Mechanics 119, 1331–1352.

Valanis, K.C., 1971. Irreversible thermodynamics of continuous media. Course held at the Department of Mechanics of Solids, Udine, Springer-Verlag, Wien-New York.

Valanis, K.C., 1990. A theory of damage in brittle materials. Engineering Fracture Mechanics 36, 403–16.

Woo, C.W., Li, D.L., 1992. A general stochastic dynamic model of continuum damage mechanics. Int. J. Solids Struct. 29, 2921–2932.

Wu, H.C., Nanakorn, C. Komarakul, 1998. An endochronic theory of continuum damage mechanics. J. of Engineering Mechanics 124, 200–208.